① intro lect 1 Jules Lamers "intro to q integr" sept 2023

Quantum integrability ⊂ exactly solvable

models from **theor phys**

w/ underlying **quantum-algebraic** structure

(rep th: q grps, ..)

→ many **symmetries**

→ **exact** analytic methods

hep-th
↳ AdS/CFT Didina Ivan
↳ $\mathcal{N}=2$ SYT
↳ ...
↳ high-E QCD Grisha

CFT Hubert
stat mech Sylvain Jesper Philippe Di F
↳ condmat
↳ nonequilib Kirone Vincent

cond mat Grégoire

historically, **main categories.**

spin chains magn	1+1d QM lattice	Heisenberg	(cf oppenh) (Mills) Bethe, Yang-Yang Faddeev "Leningrad" Jimbo-Miwa "Kyoto"	Gaudin
lattice models phase trans	2+0 d class stat mech lattice	Ising, ice-type	Onsager! Wu McCoy Lieb, Sutherland Baxter	universality & ~ classes
quantum many-body systems	1+1d QM cont	Lieb-Liniger, Calogero-Sutherland	Lieb Sutherland	
quantum field theory	1+1d QFT cont	sinh-Gordon	Zamolodchikov^2	

hist import toy models Dorey hep-th/9810026

more detail: see next page

Why quantum integrability

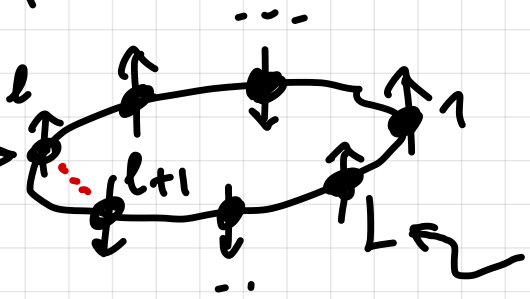
Applications

NB. I did not treat this in the lecture

stringy	AdS/CFT	worldsheet theory	integr QFT	Beisert etc etc etc
		nonpert tests		
		$\mathcal{N}=4$ SYM planar limit	~ integr spin chains	
	$\mathcal{N}=2$ SYM	susy vacua (coulomb branch) geometric rep th	~ integr spin chains	Nekrasov Shatashvili '09
	$\mathcal{N}=2$ SCFT		~ integr spin chains	Pommoni
	$\mathcal{N}=1$ SYM	Seiberg duality	~ integr lattice models	Spiridonov Yamazaki
	$\mathcal{N}=1$ SYM top twist 4d topological Chern-Simons	knot theory	~ integr models	Costello '13+ Witten Yamazaki
QFT	high-energy QCD		~ integr spin chains	Faddeev Korchemski
	2d CFT	$c<1$	$\supset$ integr model	Bazhanov Lukyanov Zamolodchikov
	historically important toy models		integr QFT	
stat mech	phase transitions, universality	algebraic combinatorics	integr lattice models	Onsager Baxter
	nonequilibrium	(T)ASEP integrable probability	~ integr spin chains	(..)
cond mat	magnetism	neutron scattering	integr spin chains	see e.g. 1307.4077
	ultracold atoms	optical traps	integr QMBS	
	FQHE	toy model	integr spin chains	Haldane Shastry

(2.1) Heis XXZ spin chain

1d crystal "atom" "site"



L length

w/ spin $\frac{1}{2}$: $|\uparrow\rangle, |\downarrow\rangle$

local Pauli's

$\sigma^{\pm} := \frac{\sigma^x \pm i\sigma^y}{2}$

$\sigma_\ell^\alpha = \mathbb{1}^{\otimes(\ell-1)} \otimes \sigma^\alpha \otimes \mathbb{1}^{\otimes(L-\ell)}$ $\begin{cases} [\sigma_k^+, \sigma_\ell^-] = \delta_{k\ell}\sigma_\ell^z \\ [\sigma_k^z, \sigma_\ell^\pm] = \pm 2\delta_{k\ell}\sigma_\ell^\pm \end{cases}$

$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\sigma^z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\mathbb{1} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

Hilb: $\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$

Ham: $H_\Delta = \boxed{-} \sum_{\ell \in \mathbb{Z}_L} h^\Delta_{\ell\,\ell+1}$

periodic BCs $h_{L\,L+1} \equiv h_{L1}$ $(= h_{1L})$

$\Delta \in \mathbb{R}$ anisotropy

nearest neighb

kinetic (hopping) potential

"a lattice gas of hardcore bosons" (can't occupy same site) i.e. $\downarrow$'s

$$h^\Delta_{\ell\,\ell+1} = \frac{1+\Gamma}{2}\left(\sigma_\ell^+\sigma_{\ell+1}^- + \sigma_\ell^-\sigma_{\ell+1}^+\right) + \Delta\,\frac{\sigma_\ell^z\sigma_{\ell+1}^z - \underbrace{1}}{2}$$

$+ \frac{1-\Gamma}{2}\left(\sigma_\ell^+\sigma_{\ell+1}^+ + \sigma_\ell^-\sigma_{\ell+1}^-\right)$ pair creation annih

sets $H_\Delta|\uparrow..\uparrow\rangle = 0$

eig val $0, 0, \Delta \pm 1$

$\Delta = 1: 0^3, -2$

$$= \begin{pmatrix} 0 & & & \frac{1-\Gamma}{2} \\ & -\Delta & \frac{1+\Gamma}{2} & \\ & \frac{1+\Gamma}{2} & -\Delta & \\ \frac{1-\Gamma}{2} & & & 0 \end{pmatrix}_{\ell,\ell+1} = \mathbb{1}^{\otimes(\ell-1)} \otimes \begin{pmatrix} \cdot & & & \\ & \cdot & \cdot & \\ & \cdot & \cdot & \\ & & & \cdot \end{pmatrix} \otimes \mathbb{1}^{\otimes(L-\ell-1)}$$

wrt $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$

models magnetic materials

- 1d – lab: quasi-1d (optical lattices, ..)
  – nature: interactions effectively 1d ($KCuF_3$, ...) (cubic) perovskite 1307.4077 PRL $\Delta = -1$
- nearest neighbour will look at >nn in lect 4-5
- periodic

goal $\begin{cases} \text{study (low-lying) spec, corr fns, ...} \\ \text{for finite but arbitrary } L, \text{ then } L \to \infty \end{cases}$ i.e. analysis

not in these lects ~ alg

intuition:

$$\lim_{\Delta \to \infty} \pm \frac{H_\Delta}{|\Delta|} = \boxed{\mp} \sum_{\ell \in \mathbb{Z}_L} \frac{\sigma_\ell^z\sigma_{\ell+1}^z - 1}{2}$$

1d Ising i.e. class stat mech: already diagonal trivial; no phase transition

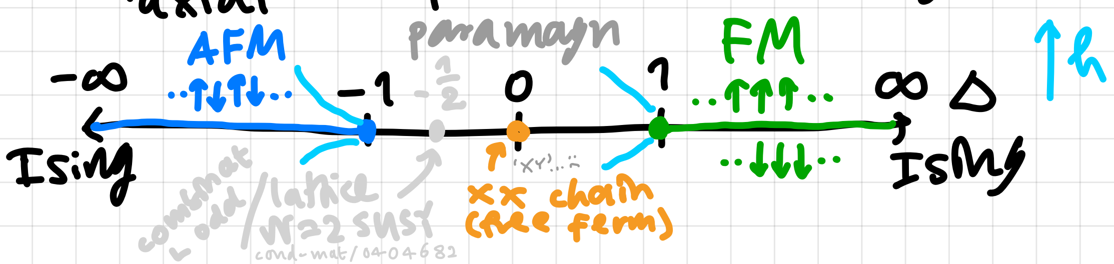
counts $..\uparrow\downarrow.., ..\downarrow\uparrow..$

sign	lowest energy	elem excitat
+	$\lvert\uparrow..\uparrow\rangle, \lvert\downarrow..\downarrow\rangle$	$..\uparrow\downarrow.., ..\downarrow\uparrow..$ (DWs) ('defects')
−	$\lvert\uparrow\downarrow..\uparrow\downarrow\rangle, \lvert\downarrow\uparrow..\downarrow\uparrow\rangle$ (L even)	$..\uparrow\uparrow.., ..\downarrow\downarrow..$

} simple in position space

phase diagr (L even, T=0)

'axial' 'planar' 'axial' regime



$\uparrow h$

[in the lect I focussed on $h=0$]

③

exc find spec $\Delta=0$ by Jordan-Wigner: $\{\sigma_\ell^+,\sigma_\ell^-\}=1$ but comm if $k\neq\ell$

$c_\ell := \sigma_1^z \cdots \sigma_{\ell-1}^z \sigma_\ell^+$

$c_\ell^+ := \underbrace{\sigma_1^z \cdots \sigma_{\ell-1}^z}_{\cdot^2=1} \sigma_\ell^-$

$n_\ell := c_\ell^+ c_\ell = \frac{1-\sigma_\ell^z}{2}$

- check CAR $\{c_k, c_\ell^+\} = \delta_{k\ell}$ $\quad \{c_k,c_\ell\}=\{c_k^+,c_\ell^+\}=0$

  & BCs $\quad c_{L+1}=(-1)^M c_1 \quad M=\sum n_\ell = \frac{L}{2}-S^z$ #op
- write H_0 (or H_Δ or $H_{\Gamma,\Delta}$) via c, c^+
- Fourier transf & read off spec $\quad c_\ell^+ = \frac{1}{\sqrt{L}} \sum_{p\in\frac{2\pi}{L}(\mathbb{Z}_L+\frac{1}{2}\delta_{M,\text{odd}})} e^{ip\ell}\, \tilde{c}_p^+$ ← also CAR

simple in mtm space: ground state = Fermi sea; excitations by moving mtm out of sea

② 2.2 symms

$H_{\Delta,h} = H_\Delta - \frac{h}{2}\sum \sigma_\ell^z$ longit^l ext^l magn field (chem^l pot)

exc i) $U=\prod_{\ell=1}^{L}\sigma_\ell^x \quad U H_{\Delta,h} U^{-1} = H_{\Delta,-h}$ global spin flip ↑ ⟷ ↓

ii) $V=\prod_{\ell=1}^{\lfloor L/2\rfloor}\sigma_{2\ell}^z$ $\quad$ L even: $V H_\Delta V^{-1} = -H_{-\Delta}$

odd w/ 'twisted' BCs $\sigma_{L+1}^\alpha = \sigma_1^z \sigma_1^\alpha \sigma_1^z = \begin{cases} -\sigma_1^\pm & \alpha=\pm \\ \sigma_1^z & \alpha=z\end{cases}$

partial isotropy $U(1)$: $S^z := \frac{1}{2}\sum_{\ell=1}^{L}\sigma_\ell^z$ magnetiz^n

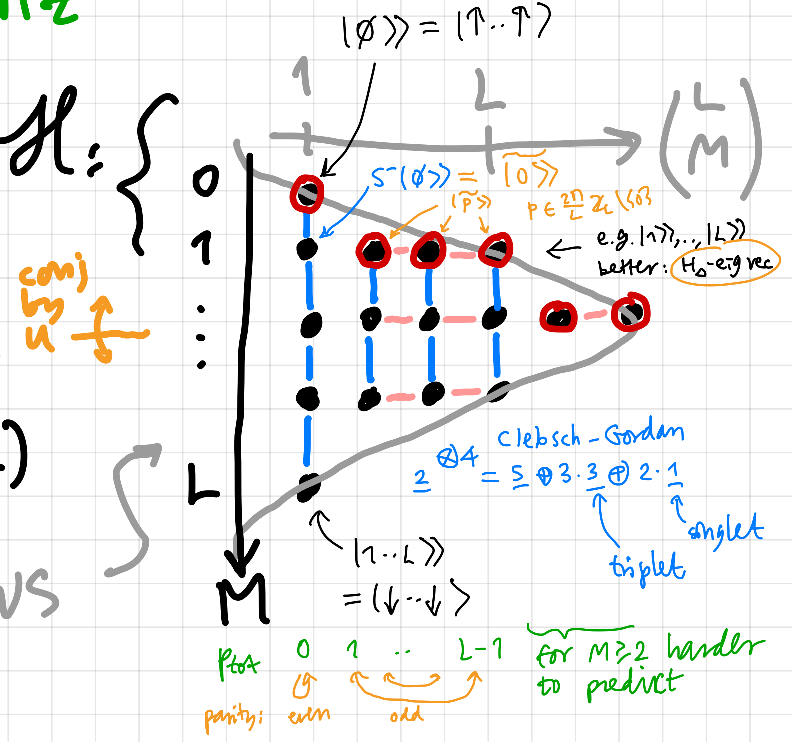
$[S^z, H_\Delta]=0 \quad \forall\Delta$ $\quad (\Gamma=1)$!!!

$\Rightarrow$ diagonalise H_Δ per $M=\frac{L}{2}-S^z = \#\downarrow$

'coordinate basis' $|\ell_1,\ldots,\ell_M\rangle\rangle := \sigma_{\ell_1}^-\cdots\sigma_{\ell_M}^-|\uparrow..\uparrow\rangle$ $\quad (1\leq \ell_1<\ldots<\ell_M\leq L)$

$\mathcal{H}=\bigoplus_{M=0}^{L}\mathcal{H}_M$

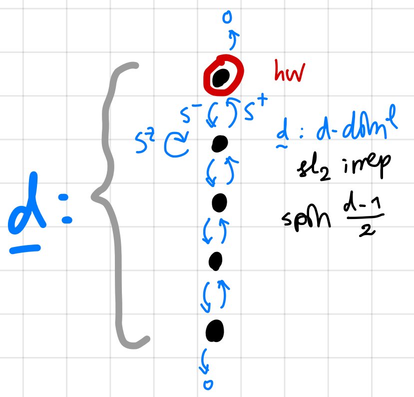
i.e. H_Δ block diag w/ $\binom{L}{M}\times\binom{L}{M}$ blocks, ~rows



global spin flip: $M\leq L/2$ 'equator'

$H_\Delta|\emptyset\rangle\rangle = 0$ first eig vec

isotropic point: $S^\pm = \sum_{\ell=1}^{L}\sigma_\ell^\pm$ su(2) ladder operators $\quad [S^+,S^-]=2S^z \quad [S^z,S^\pm]=\pm S^\pm$



$[S^\pm, H_\Delta]=0$ at $\Gamma=\Delta=1$

in this case all ●'s conn by blue lines have same energy

so suffices to find $|\psi\rangle$ s.t. $S^+|\psi\rangle=0$ (h.w.)

e.g. $S^-|\uparrow..\uparrow\rangle \propto \sum_{\ell=1}^{L}|\ell\rangle\rangle$ also $E=0$ 'Goldstone mode' if $\Delta=1$

infinitesimal rot^n of magnetism

NB. $h^{\Delta=1} = \begin{pmatrix} 0 & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & 0\end{pmatrix} = 1-P, \quad P=\begin{pmatrix}1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1\end{pmatrix}$ two-site spin permutation 'exchange interaction'

∝ antisymm projector onto singlet $\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{2}$

homogeneity: $G|\ell_1,\dots,\ell_M\rangle\rangle = |\ell_1 - 1 \bmod, \dots, \ell_M - 1\rangle\rangle$ cyclic (left) translation

$$G H_\Delta G^{-1} = H_\Delta \quad \forall \Delta$$

$G^L = 1$ so eig val $e^{i p_{tot}}$ **momentum** $p_{tot} \in \frac{2\pi}{L}\mathbb{Z}_L$ (mod 2π)

Fixes $M=1$:

exc check $|\widetilde{p}\rangle\rangle = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} e^{ip\ell} |\ell\rangle\rangle$ have $p_{tot} = p$

Fourier transf of $|\ell\rangle$ ← plane wave

- orthog
- $\# = \dim \mathcal{H}_{M=1}$
- energy: see below

$\langle \ell | G | \psi \rangle$
$= \langle \ell+1 | \psi \rangle$
$= \psi(\ell+1)$
$\overset{!}{=} e^{ip}\psi(\ell)$

parity: $\Pi|\ell_1,\dots,\ell_M\rangle = |L-\ell_M+1,\dots,L-\ell_1+1\rangle$

$$\Pi H_\Delta \Pi^{-1} = H_\Delta \ \forall\Delta$$

$$\Pi G \Pi^{-1} = G^{-1}$$ so parity relates $\pm p_{tot}$

- Π-singlet even : $p_{tot} = 0, \pi$
- Π-doublet odd : $\pm p_{tot}$

$u(1)_z$ + transl fixes $M=0,1$
but not $2 \leq M \leq L-2$

have to be smart: follow Bethe

coord Bethe ansatz:

physically meaningful param^n of would-be eig vecs; are true eig vec if param^s obey "BAE"

next time: gain from doing this

rmk rep th: Schur-Weyl $\mathbb{C}^2 = V_\square$ fund rep $\mathfrak{sl}_2$

structure of $\mathcal{H}$ (may or may not be preserved by Ham)

$\mathfrak{h} \quad \bigoplus \mathcal{H}[L-2M]$ wt spaces

$\cap \quad \mathfrak{h} \, ||$

$\langle U \rangle =$ Weyl grp $\circlearrowright$ $\mathfrak{sl}_2 \circlearrowright \mathcal{H} \circlearrowleft S_L \supset \mathrm{Dih}_L = S_2 \ltimes \mathbb{Z}_L$
coprod

Dih_L: $S_2 = \langle \Pi \rangle$, $\mathbb{Z}_L = \langle G \rangle$

(ii) H_Δ only inv^t under $\mathrm{Dih}_L = S_2 \ltimes \mathbb{Z}_L$

decomp of $\mathcal{H}$ for $\mathfrak{sl}_2$, Dih_L has mult at $2 \leq M \leq L-2$

$||$ ← mult free as bimod

dim $L-2M_0+1$ irrep $\lambda = (L-M_0, M_0)$ $\mathfrak{gl}_2$
$\cong (L-2M_0)$ $\mathfrak{sl}_2$

$$\bigoplus_{\substack{\lambda \vdash L \\ \ell\lambda \leq 2}} V_\lambda \otimes S^\lambda$$

irrep: Specht mod
$\dim = \# SYT(\lambda)$
$= \binom{L}{M} - \binom{L}{M-1}$

(i) this decomp would be relevant for 'mean field' $\sum h_{k\ell}|_{\Delta=1}$ quadr casimir (vs DWave)

NB. I do not mean to suggest that a integ would be precisely use of Schur-Weyl; but at least in some cases it does

NB. $\circlearrowleft S_L$ not faithfull
↓
$\circlearrowleft TL_L(2)$
faithful

$e_\ell = -h_{\ell\ell+1}^{\Delta=1} = 1 - P_{\ell\ell+1}$ obeys
$1 \leq \ell < L$

$$\begin{cases} e_k e_{k\pm1} e_k = e_k \\ e_k^2 = 2 e_k \\ [e_k, e_\ell] = 0 \quad |k-\ell| > 1 \end{cases}$$

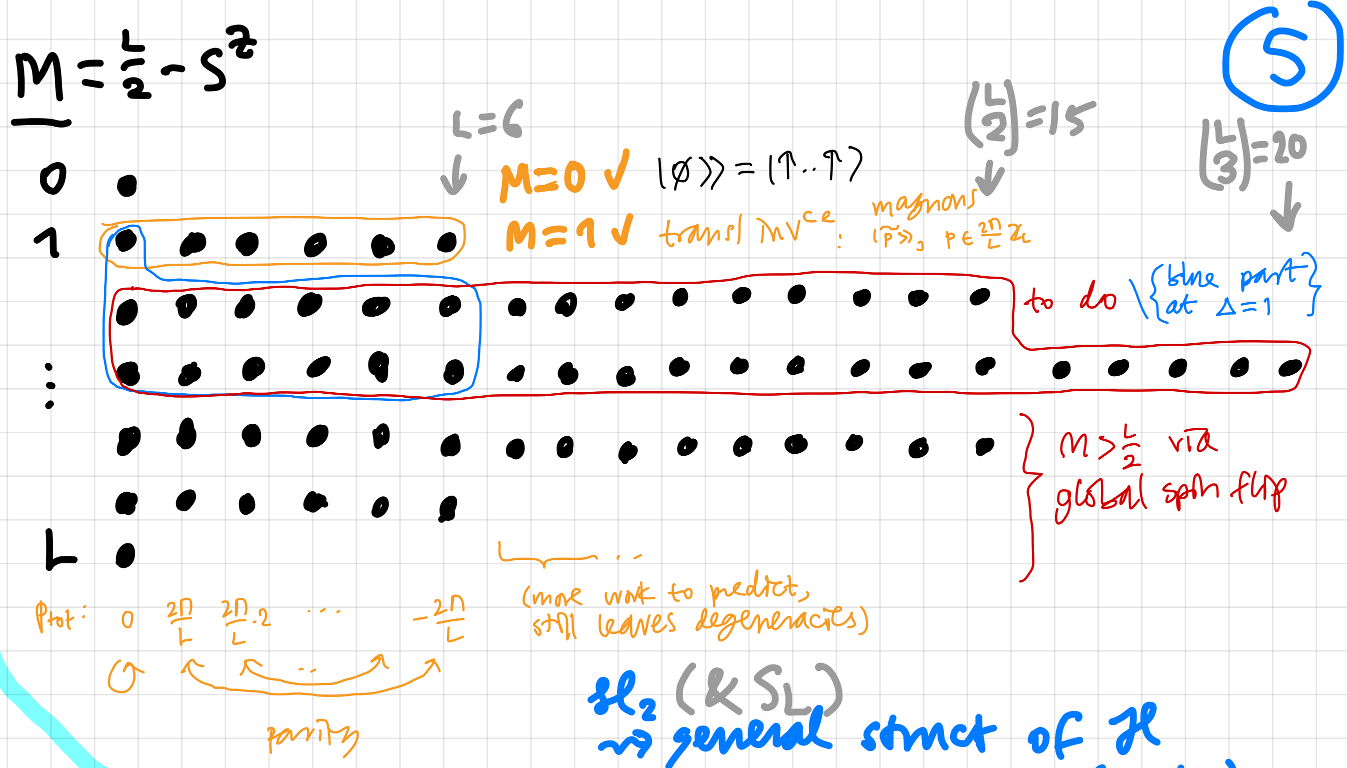
recap lect 2

$\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$:

G transl: $p_{tot} \in \frac{2\pi}{L}\mathbb{Z}_L$

$$H_\Delta = -\sum_{\ell\in\mathbb{Z}_L} h_{\ell\ell+1}$$

$$h_{\ell\ell+1} = (\sigma^+_\ell \sigma^-_{\ell+1} + \sigma^-_\ell \sigma^+_{\ell+1}) + \Delta(\sigma^z_\ell \sigma^z_{\ell+1} - 1)/2$$



sl_2 (& S_L) $\rightsquigarrow$ general struct of $\mathcal{H}$

<u>not</u> H_Δ-eig vec ($\Delta \neq 1$)

except for $S^- |\emptyset\rangle\rangle = |\tilde{0}\rangle\rangle$

③ coordinate Bethe ansatz (CBA) old school integrab

physically meaningful param^n of would-be eig vec^s

will see: true eig vecs if param^s obey "BAE"

3.1 preliminaries

(*) $H_\Delta|\psi\rangle = E_\Delta|\psi\rangle$, fix $M = \frac{L}{2} - S^z = \#\downarrow$

notation: $|\underline{\ell}\rangle\rangle := |\ell_1, \dots, \ell_M\rangle\rangle$ $\downarrow$'s @ $1 \leq \ell_1 < .. < \ell_M \leq L$

$|\underline{\ell} \pm 1\rangle\rangle := |\ell_1 \pm 1, .., \ell_M \pm 1\rangle\rangle$ (mod L : $\sigma^\alpha_{L+1} = \sigma^\alpha_1$)

a priori, Ψ defined for 'simplex' (weyl alcove) $1 \leq \ell_1 < .. < \ell_M \leq L$; but hard to do book keeping to account for per.BCs (1 & L also adjacent)

periodicity: $|\ell_1, .., \ell_M\rangle\rangle = |\ell_2, .., \ell_M, \ell_1 + L\rangle\rangle$ ext'd to $\ell_1 < .. < \ell_M$ s.t. $\ell_M - \ell_1 < L$

then $\Psi(\underline{\ell}) = \langle\langle \underline{\ell}|\psi\rangle$ wave f^n must be periodic

& obey 2nd order difference eq^n w/ const coeff^s

$$\sum_{m=1}^{M} \sum_{k=1}^{L}{}' \Psi(\underline{\ell})|_{\ell_m \mapsto k} \overset{(\$)}{=} (-E_\Delta + 2\overbrace{(M - N_{\underline{\ell}})}^{\#\text{DWs}}\Delta)\, \Psi(\underline{\ell})$$

(= # allowed hoppings)

$k \in (\underline{\ell}-1) \cup (\underline{\ell}+1) \setminus \underline{\ell}$

hopping to unoccupied neighbour

$\# \underline{\ell} \cap (\underline{\ell}+1)$

neighbours in $\underline{\ell}$ (mod L)

exc $M=1$: $\boxed{\psi(\ell+1) - 2\Delta\psi(\ell) + \psi(\ell-1)} = -E_\Delta \psi(\ell)$

(how see isotropy at Δ=1?)

$\psi(\ell) = e^{ip\ell}$ $\Rightarrow$ $\varepsilon_\Delta(p) = E^{M=1}_\Delta = 2(\Delta - \cos p)$

Δ=1: discr laplacian (geom interpret Δ≠1?)

dispersion relation

pot^l

kinetic

this diagonalises L×L block of H_Δ with M=1

exc check (*) gives (\$)

(3.2) $M=2$

⑥

a) $N_{\underline{\ell}}=0$: $\Psi(\ell_1+1,\ell_2)+\Psi(\ell_1,\ell_2+1)+\Psi(\ell_1-1,\ell_2)+\Psi(\ell_1,\ell_2-1)-4\Delta\Psi(\ell_1,\ell_2)=-E_\Delta\Psi(\ell_1,\ell_2)$

"well separated"

b) $N_{\underline{\ell}}=1$: $\Psi(\ell_1-1,\ell_2)+\Psi(\ell_1,\ell_2+1)-2\Delta\Psi(\ell_1,\ell_2)=-E_\Delta\Psi(\ell_1,\ell_2)$

$\ell_1+1=\ell_2$ adjacent ← incl $n_1=1$, $n_2=L$ via periodic BCs

simplify (b) using (a) ?!

$(\sigma^-)^2=0 \Rightarrow |..,\ell,\ell,..\rangle\rangle=0$

nonphys

assume can (formally) extend Ψ to $\ell_1\leq\ldots\leq\ell_M$, $\ell_M-\ell_1<L$

if so, suffices to consider

- $(b')=(a)_{\substack{n_1=n\\n_2=n+1}}-(b)$: $\Psi(\ell,\ell)+\Psi(\ell+1,\ell+1)-2\Delta\Psi(\ell,\ell+1)=0$ "bounce condn"
- (a) now extended to all ℓ_1,ℓ_2 "wave eqn"
- periodic BCs $\Psi(\ell_1,..,\ell_M)=\Psi(\ell_2,..,\ell_M,\ell_1+L)$ (c)

(b') & (c) cut out 'wrong' extd solns from space $\{\psi:(b)\}$

ad b) note: $(M=1)_{\ell=\ell_1}+(M=1)_{\ell=\ell_2}=-E_\Delta\psi$ separation of variables

⇒ factorised sol $\psi=e^{ip_1\ell_1}e^{ip_2\ell_2}=e^{i\underline{p}\cdot\underline{\ell}}$ $\underline{p}=(p_1,p_2)$

plane wave

$\Rightarrow E_\Delta(\underline{p})=\varepsilon_\Delta(p_1)+\varepsilon_\Delta(p_2)$

can't be right; not free: interactions for adj ↓'s

but (b') only solved by $p_1=p_2=0$...
need to account for interactions

Bethe ansatz: parametrise by $\underline{p}=(p_1,p_2)$

$$\psi_{\underline{p}}(\ell_1,\ell_2)=A(\underline{p})\underbrace{e^{i\underline{p}\cdot\underline{\ell}}}_{\text{plane wave}}+A'(\underline{p})\underbrace{e^{i\underline{p}'\cdot\underline{\ell}}}_{\text{(classically) scattered wave}}$$

$\underline{p}'=(p_2,p_1)$

assn ok (no poles or so) ← (unlike for Frozemtsev)

7

(b) still OK

(b'): $\frac{A}{A'} = -\frac{1-2\Delta e^{ip_1}+e^{i(p_1+p_2)}}{1-2\Delta e^{ip_2}+e^{i(p_1+p_2)}} =: S_\Delta(p_1,p_2)$ '2-magnon S-matrix' (1×1)

indept of ℓ

(c): $e^{ip_1L} = S(p_1,p_2)$
$e^{ip_2L} = S(p_1,p_2)^{-1}$ BAE $\Big\} \Rightarrow e^{i\underbrace{(p_1+p_2)}_{p_{tot}}L} = 1$

exc work out details M=2

exc repeat M=3 (masochists)

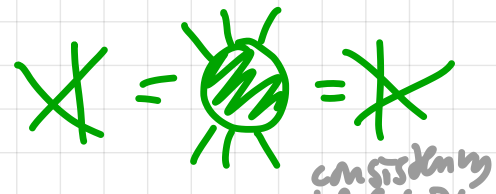
(3.3) general M

CBA $\Psi_p(\ell) = \sum_{w\in S_M} A_w(p)\, e^{i\boxed{p_w}\cdot\ell}$ $(p_{w1},\dots,p_{w(M)})$

all perms of magnons

proceed as before:

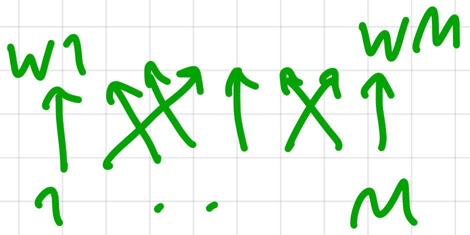
(b) OK, $E_\Delta(p) \boxed{=} \sum_{m=1}^{M} \varepsilon_\Delta(p_m)$ (fnct'ly) additive energy

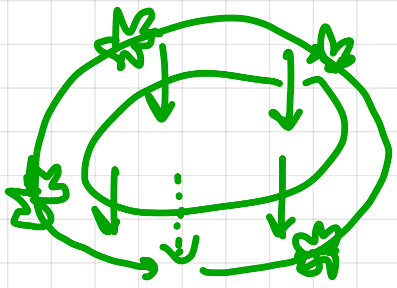

note $\times = \circledast = \times$

consistency here triv (1×1 matrix)

b') $\rightsquigarrow$ $\frac{A_w}{A_e} = \prod_{\substack{m<n \\ \text{s.t. } w m > w(n)}} S_\Delta(p_m,p_n)^{-1}$ inversion of w

factorized scattering (2-magnon reducible) (⊥ free)





(c) $e^{ip_mL} = \prod_{n(\neq m)} S_\Delta(p_m,p_n)$ BAE

quantn condns for p_m

coupled: interactions (scattering)

$\rightsquigarrow e^{ip_{tot}L} = 1$

$\sum p_m \boxed{=} p_{tot} \in \frac{2\pi}{L}\mathbb{Z}_L$

quasi momenta

quasiparticle picture

scattering phase shift

$\varphi_\Delta(p_1,p_2) := -i \log S_\Delta(p_1,p_2)$ $\cot\frac{\varphi_\Delta(p_1,p_2)}{2} = \frac{\Delta \sin\frac{p_1-p_2}{2}}{\cos\frac{p_1+p_2}{2} - \Delta\cos\frac{p_1-p_2}{2}}$

log BAE: $p_m = \underbrace{\frac{2\pi}{L} I_m}_{\text{'bare' (free)}} + \underbrace{\frac{1}{2L}\sum_{m'(\neq m)}^{M} \varphi_\Delta(p_m,p_{m'})}_{\text{interactions } (\propto\Delta)}$

$I_m \in \mathbb{Z}_L$ Bethe QNs

exc $\Psi_{\underline{p}} = 0$ if $p_m = p_{m'}$ $(m \neq m')$ Pauli principle

'completeness': this gives all eig vec   Mukhin TV 09

to do: find all sol^ns to BAE corresp to lin indep eig vec

exc solve for $\Delta = 0$

exc solve for $\Delta \to \pm\infty$ assuming $p_m \in \mathbb{R}\ \forall m$

in general hard, but still useful: e.g.

skipped in lecture

$\Delta = 1$: • act by S^- ⇔ add $p_0 = 0$ to sol^n   descendant

$(I_0 = 0,\ \varphi_{\Delta=1}(0, p_n) = 0)$

can show $p_m \neq 0$ mod 2π $\forall m \Rightarrow S^+ \sum \Psi_{\underline{p}}(\underline{\ell}) |\underline{\ell}\rangle\rangle = 0$ hw $S_{\Delta=1}$ discont at (0,0)

- $\forall\ 1 \le I_1 < \dots < I_M \le L-1$ s.t. $I_{m+1} > I_m + 1$ well separated

  $\exists$ sol^n w/ $p_m \approx \frac{2\pi}{L} I_m$ real   scattering states

- (almost) all other sol^ns contain some $\operatorname{Im} p_m \neq 0$

  $E_\Delta(\underline{p}) \in \mathbb{R} \Leftrightarrow \overline{\{p_1, \dots, p_M\}} = \{p_1, \dots, p_M\}$ bound states

  cpx conj pairs   ".. ↓↓ .." true for $\Delta \to \infty$

conj for combinatorics: "string hyp" Takahashi

strictly speaking false

but suff't to determine exact thermodyn $L \to \infty$

$\Delta < 1$: AFM ground state Yang-Yang '66

L even, $M = L/2$ equator, $\underline{I} = (1, 3, 5, \dots, L-1)$, $\overset{\substack{\mathbb{R}\\ \cup}}{p_m} \approx \frac{2\pi}{L} I_m$

'fermi sea'

i.e. Perron-Frobenius vector   cf e.g. 2309.02008

calculation bit painful; rite of passage for integrals

but hope main ideas clear & esp phys results

takeaway message: quasiparticles 'very polite': lots of rules of engagement

\# magn preserved, fact scatt: hidden symm ?? → integrab

next: ice model (2d class stat mech)

will see: closely related

& helps discover hidden symm

\+ much easier alt to CBA

Feynman final blackb

"I got really fascinated by these 1+1 d models that are solved by Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better."

3.4 comments

I only mentioned part of this

a) (phys) unitarity $|S_\Delta(p_1,p_2)| = 1$ if p_1, p_2 real

b) (braiding) 'unitarity' $S_\Delta(p_1,p_2)^{-1} = S_\Delta(p_2,p_1)$ $\therefore$ φ_Δ antisymm

normalise

$$A_e = \prod_{m<n}^{M} S(p_m,p_n)^{1/2} = e^{\frac{i}{2}\sum_{m<n}^{M}\varphi_\Delta(p_m,p_n)}$$

$$\therefore \Psi_p = \sum_{w\in S_M} \exp\left(i p_w \cdot \underline{n} + \frac{i}{2}\sum_{m<n}^{M}\bar{\varphi}_\Delta(p_{wm},p_{wn})\right)$$

formally symm (like coord basis §3)

c) $S_\Delta(p_1,p_2)|_{p_1 \to p_2} = -1$ ~ fermionic property ; $S_{\Delta=1}(p_1,0) = 1$ discont at $p=0$ for $\Delta = 1$!

exc $\Psi_p = 0$ if $p_m = p_{m'}$ $(m \neq m')$ unless $\Delta = 1$, $p_m = p_{m'} = 0$

'pauli principle' for quasimomenta ($\Delta = -1$ $p_m = p_{m'} = \pi$ L even

d) $\Delta = 1$: can show $S^+ \sum \Psi_p(\underline{\ell})|\underline{\ell}\rangle\!\rangle = 0$ if $(p_m \neq 0 \bmod 2\pi\ \forall m)$

$\varphi_\Delta(p_1,p_2)|_{p_1\to p_2} = \pi$ some authors absorb $\varphi_\Delta \to \varphi_\Delta - \pi =: \bar{\varphi}_\Delta$ $\rightsquigarrow$ explicit $\mathrm{sgn}(w)$ in CBA

d) rapidities $\Delta = \cosh(\eta)$

$$e^{ip_m} = \frac{\sin(\lambda_m + i\eta/2)}{\sin(\lambda_m - i\eta/2)} \qquad \Delta = 1:\ \lambda = \frac{1}{2}\cot\frac{p}{2}$$

$$S_\Delta(p,p') = \frac{\sinh(\lambda - \lambda' + i\eta)}{\sinh(\lambda - \lambda' - i\eta)} \quad f^n \text{ of diff}^{ce}$$

$$\varepsilon_\Delta(p_m) = \frac{\sinh^2\eta}{\sinh(\lambda_m + i\eta/2)\sinh(\lambda_m - i\eta/2)}$$

exc compute $p'(\lambda_m)$ & compare w $\uparrow$

(we'll understand in lect 3)

④ six-vertex model

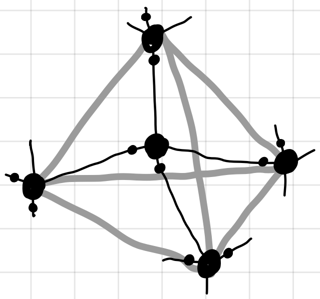
4.1 ice & friends

ordinary 'Ih' ice:

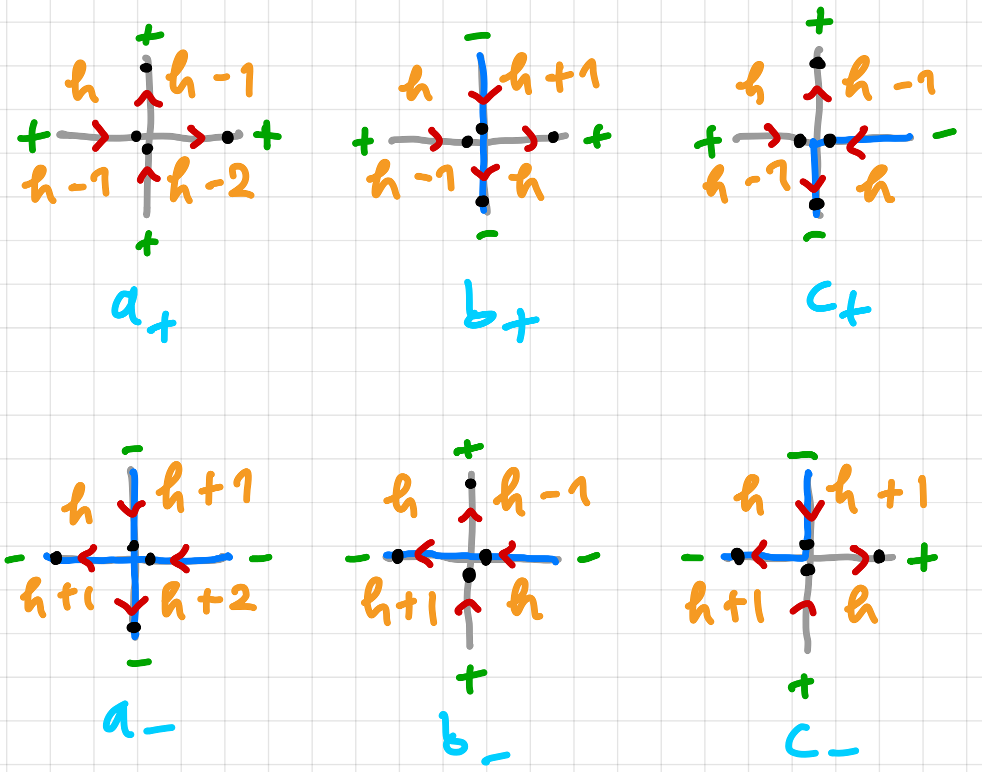
hexagonal crystal

H_2O molecules

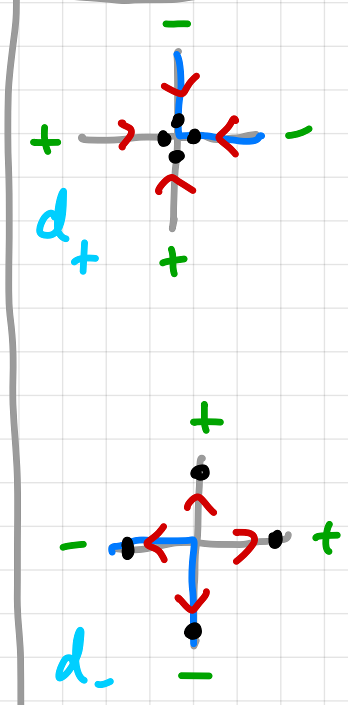
elec neutrality: one proton per H bond
two H's near each O



2d approx: square lattice



NB. 8V



'charge defects'

sink / source

path conservation
without orientation

'spin' pair annih/
creation

not OK

stat mech: 6v (ice-type) model

vx model ↓

$$E(\text{config}) = \sum_{\text{vertices}} \varepsilon_{\text{vertex}}\left(\begin{smallmatrix}\text{config}\\ \text{around}\\ \text{vertex}\end{smallmatrix}\right)$$

canonical part^n f^n $\qquad \beta := \frac{1}{k_B T}$

$$Z = \sum_{\text{configs}} e^{-\beta E(\text{config})} \text{ ⊜}$$

$$\text{prob}(\text{config}) = \frac{1}{Z} e^{-\beta E(\text{config})}$$

vertex weights $a_+ = e^{-\beta \varepsilon_{a_+}}$ etc

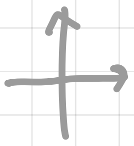
$a_\pm, b_\pm, c_\pm \geq 0$

we'll focus on 'zero field' $\left\{\begin{array}{l} a_\pm = a \\ b_\pm = b \\ c_\pm = c \end{array}\right.$

$$\text{⊜} \sum_{\text{config}} a^{\#a's} b^{\#b's} c^{\#c's}$$

local d.o.f (on edges)	local constraint (at vertex) 'ice rule'
proton config	electric neutrality
arrow config	divergence free Eulerian graph
'SW→NE' path config	path (line) conservation
class 'spin' ~ Ising config	'spin' (charge) conservation
height config on faces	well defined

orient lines of lattice, say



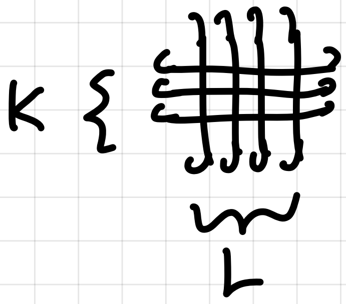
I skipped this

mapping to height (solid-on-solid) model (BCC crystal)

'reference' height @ one face
⇒ full height config via $h \underset{\pm}{\overset{\pm}{\uparrow}} h \mp 1$

goal: compute Z

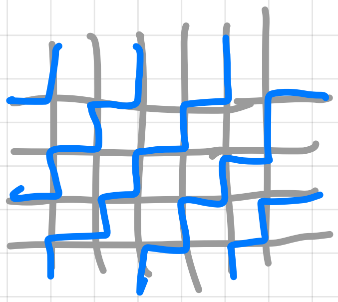
we'll consider $K \times L$ lattice on torus (periodic BCs)



special cases

- $a=b=c$ ice or $T=\infty$
- $a>b=c$ (or $b>a=c$) Slater KDP potassium dihydrogen phosphate KH_2PO_4 (ferroelectr.) polarised
- $a=b<c$ Rys F-model (antiferro electric) staggered

exc K, L even & $a, b \ll c$:
two ground states (chequerboard of $c_\pm$'s)
focussing on one of those,
show



$$\frac{1}{2} Z = 1 + N a^2 b^2 + N a^2 b^2 (a^2 + b^2) + \frac{1}{2} N(N+1) a^4 b^4 + N a^2 b^2 (a^4 + b^4) + \cdots$$

$N := KL$, wlog $c \equiv 1$

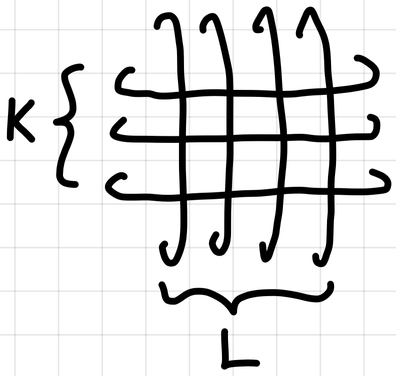
hint: draw (paths or heights mod 2)

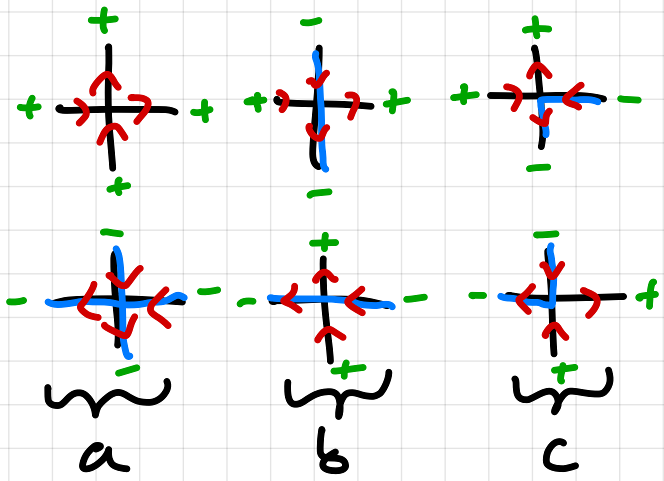
recap 6vm

4.1

goal: compute

$$Z = \sum_{\text{configs}} a^{N_a} b^{N_b} c^{N_c}$$





ice rule = path conserv.

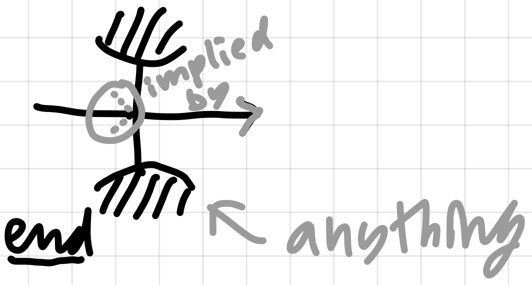
11

---

lect 3

diagrammatics:

i) orient rows & cols of lattice (usually upwards & to right)
indicate by little arrows at end



path iff ← goes against orientation:

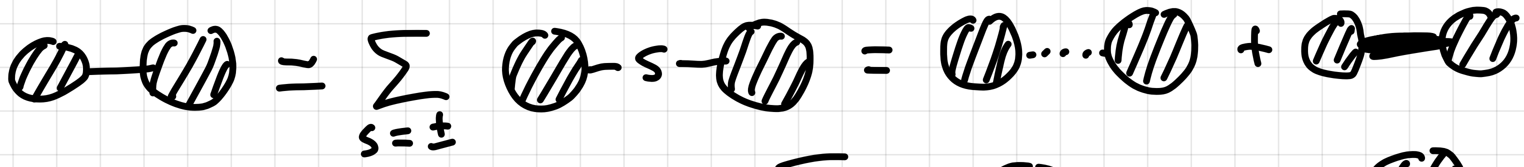
ii) 'spins'

iii) identify vx weights with their diagram:

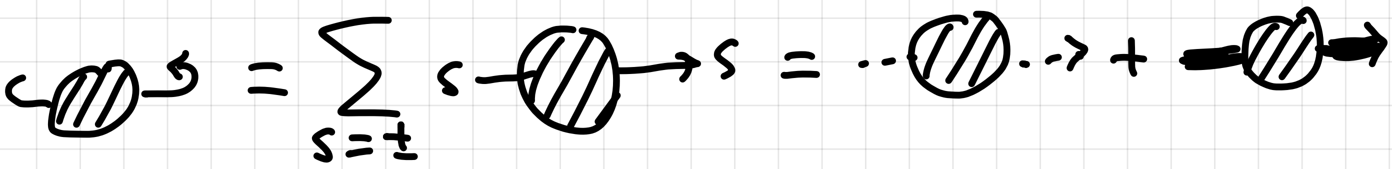
$= a = e^{-\beta \varepsilon_a}$ etc

$= 0$ (not allowed, $\varepsilon = \infty$)

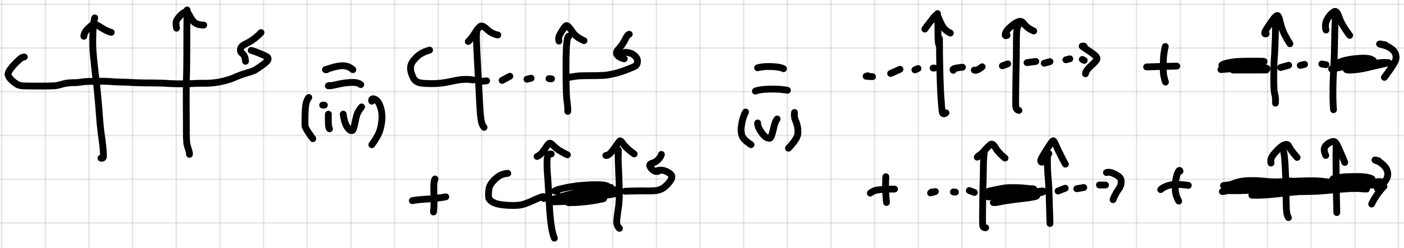
iv) summation convention for internal edges



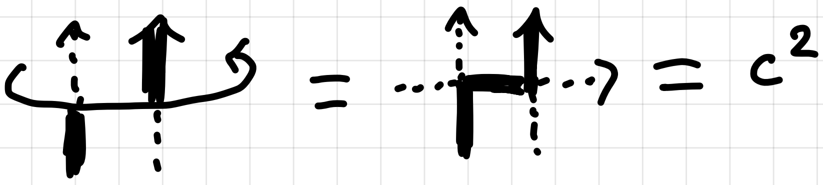
v) periodic BCs:



ex



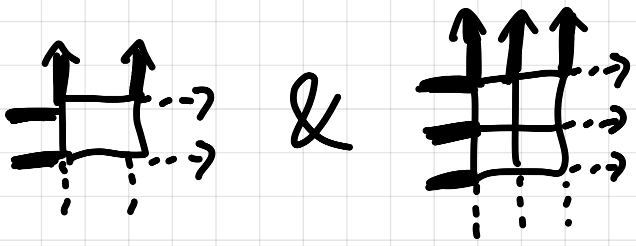
$= ba + ab = 2ab$



$= c^2$

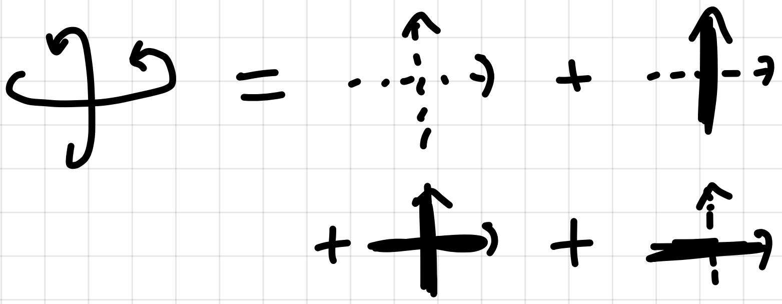
$= 0$

exc compute



graphically

ex

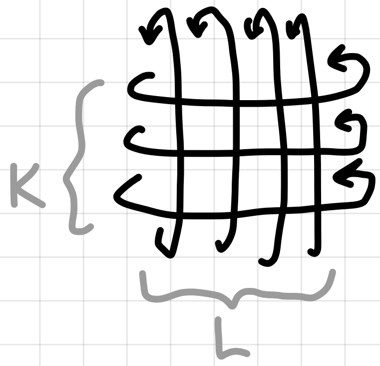


$= 2(a+b)$

⑫

exc

convince yourself:



$= \sum_{\text{spin config}} a^{N_a} b^{N_b} c^{N_c} = Z$

all edges internal ⇒ sum over $s = \pm$ for each edge

graph not'n extremely convenient; I'll use it a lot!

(4.2) transf matrix

used by Onsager for 2d Ising '44

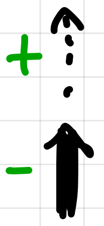
idea: write Z using lin alg

to get there we'll cut up Z into simpler smaller pieces: first into horizontal slices

associate

edge $\uparrow \longleftrightarrow \mathbb{C}^2$ v.s. then $\underbrace{\uparrow\uparrow\cdots\uparrow}_{L} \longleftrightarrow (\mathbb{C}^2)^{\otimes L}$

local d.o.f.

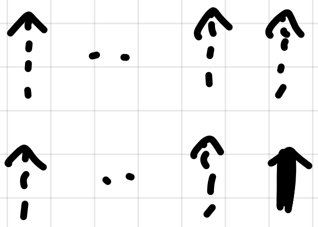


$\longleftrightarrow |+\rangle$

$\longleftrightarrow |-\rangle$

basis

e.g.



$\longleftrightarrow |+ + \cdots +\rangle$

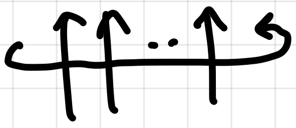
$\longleftrightarrow |+ \cdots + -\rangle$

etc

$|s_1,\dots,s_L\rangle$

transfer matrix:

$t =$

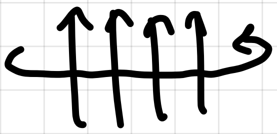


view as operator: $$\begin{array}{c}(\mathbb{C}^2)^{\otimes L}\\ \uparrow\\ (\mathbb{C}^2)^{\otimes L}\end{array}$$

matrix elt:

$\langle s_1',\dots,s_L'|t|s_1,\dots,s_L\rangle =$

$s_1' \cdots s_L'$



$s_1 \cdots s_L$

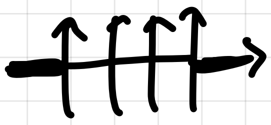
$=$

$\underline{s}'$

$\underline{s}$

$+$

$\underline{s}'$



$\underline{s}$

note: polyn in a,b,c
of tot deg L, $\leq 2^L$ terms

luckily, as we'll see, actually $\ll 2^L$

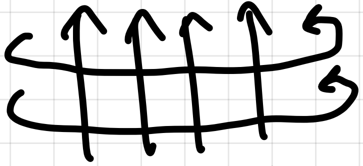
it transfers config through row:
(from below to above the row)

$$t|\underline{s}\rangle = \sum_{s_1',\dots,s_L'=\pm} \underbrace{\langle \underline{s}'|t|\underline{s}\rangle} \cdot |\underline{s}'\rangle$$

~likelihood (weight) of finding $\underline{s}'$ one row above $\underline{s}$

two rows:

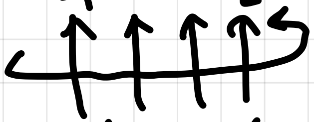
$s_1'' \cdots s_L''$



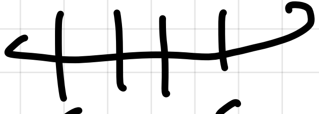
$s_1 \cdots s_L$

$= \sum_{\substack{s_1',\dots,s_L'\\ =\pm}}$

$s_1'' \cdots s_L''$



$s_1' \cdots s_L'$

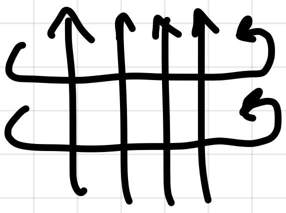


$s_1 \cdots s_L$

$$= \sum_{\underline{s}'} \langle \underline{s}''|t|\underline{s}'\rangle\langle \underline{s}'|t|\underline{s}\rangle$$

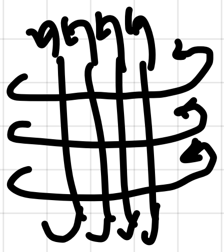
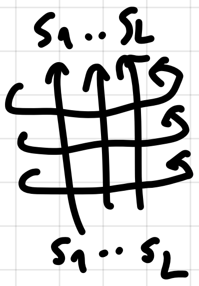
$$= \langle \underline{s}''|t^2|\underline{s}\rangle$$

so



$= t^2$ equality of operators on $(\mathbb{C}^2)^{\otimes L}$

more generally: concatenation ~ prod of op

$Z =$  $= \sum_{s_1,\dots,s_L}$  $= \sum_{\underline{s}} \langle \underline{s} | t^K | \underline{s} \rangle = \mathrm{tr}(t^K)$

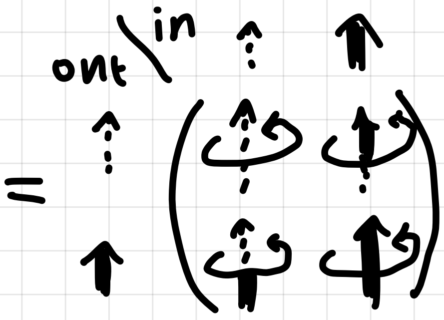
ex
- use arrow picture to check how vx weights $a_\pm, b_\pm, c_\pm$ change under reflections ↕ and ↔
- use graphical not^n to check that in zero-field case $t^T = t$ (symm) ← oops: requires parity as well?
- check that $t^* = t$ (real)

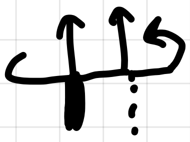
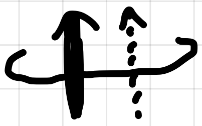
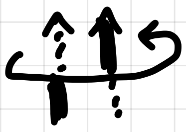
so real eig val $\Lambda_1 \geq \Lambda_2 \geq \dots$

if $\Lambda_1 > \Lambda_2 \geq \dots$ (gap), $Z = \mathrm{tr}(t^K) = \Lambda_1^K \left(1 + \underbrace{\left(\frac{\Lambda_2}{\Lambda_1}\right)}_{<1}^K + \dots\right)$

new goal: diagonalise t

so let's take closer look at t

ex $L=1$: $t =$  $= (a+b)\,\mathbb{1}$

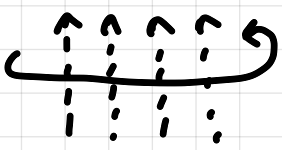
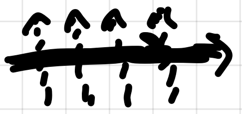
ex $t|-+\rangle =$  $=$  $\times |-+\rangle +$  $\times |+-\rangle$ } above ex

path conservation & horiz PBCs

$= 2ab\,|-+\rangle + c^2\,|+-\rangle$

exc $L=2$ check $t = \begin{pmatrix} a^2+b^2 & & & \\ & 2ab & c^2 & \\ & c^2 & 2ab & \\ & & & a^2+b^2 \end{pmatrix}$ block diag no coincidence!

note: ice rule & PBCs } ⇒ t conserves #↑'s

ex #↑=0:  $=$  $+$  $= a^L + b^L$

Lieb (sq ice, KDP, F-model), Sutherland (6vm): diagonalise t using CBA!

properties of H_Δ (vs t):

- $[S^z, H_\Delta] = 0 \Rightarrow \# \downarrow$ conserved; essential for CBA

  identify

6v	6v v.s.	Heis v.s.
$\vdots \leftrightarrow$	$+ \equiv$	$\uparrow$
$\Uparrow \leftrightarrow$	$- \equiv$	$\downarrow$

  (so $\sigma^z |\pm\rangle = \pm |\pm\rangle$)

  $\Rightarrow$ use coord basis $|\underline{\ell}\rangle\rangle$ coord of thick lines

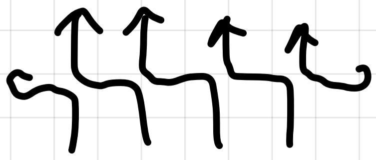
- $G H_\Delta G^{-1} = H_\Delta$ ($\Rightarrow$ momentum conserved)

  useful for eff^t bookkeeping

  PBC: $G^L = 1$, $|\ell_1, .., \ell_M\rangle\rangle = |\ell_2, .., \ell_M, \ell_1 + L\rangle\rangle$

  essential: $\rightsquigarrow$ BAE

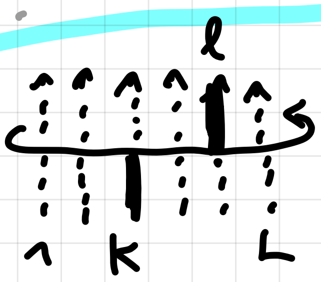
  ok w/ t:

  exc check t commutes w/ $G =$ 

- nearest neighb: diff^{ce} eq^n fairly simple

  ($\Rightarrow$ eigenvalues simple expression)

  not so for t:

  exc compute 

  $$= \begin{cases} \frac{c^2}{ab} b^L \left(\frac{a}{b}\right)^{|k-\ell|} & k \neq \ell \\ ab(a^{L-1} + b^{L-1}) & k = \ell \end{cases}$$

  (distinguish $k<\ell$, $k=\ell$, $k>\ell$)

  skipped

..but still fairly simple:

exc check $\langle \underline{\ell}' | t | \underline{\ell} \rangle$:

and compute nonzero entries

$$\begin{cases} 2 \text{ terms } \underline{\ell} = \underline{\ell}' \\ 1 \text{ term if } \underline{\ell}, \underline{\ell}' \text{ interlace: } \\ \quad \left(\begin{array}{l} \ell'_m \leq \ell_m \ \forall m \\ \vee\ \ell_m \leq \ell'_m \ \forall m \end{array} \right) \\ = 0 \quad \text{else} \end{cases}$$

& at least one strict inequality

$t |\tilde{\psi}\rangle = \Lambda |\tilde{\psi}\rangle$

strategy:

- apply $\ll \underline{\ell}|$
- CBA $\langle \underline{\ell} | \tilde{\psi} \rangle = \tilde{\psi}_{\tilde{\underline{p}}}(\underline{\ell}) = \sum_{w \in S_M} \tilde{A}_w(\tilde{p}) e^{i\tilde{p}_w \cdot \underline{\ell}}$ ↖ now coord of ▮'s
- for LHS combine terms via geom sums

solve.

- 'wanted terms' $\propto e^{i\tilde{p}_w \cdot \underline{\ell}}$ coeff $= 0 \Rightarrow \Lambda = \Lambda(\tilde{p})$
- 'unwanted terms' (rest)
  - 'internal' $\propto e^{i(p_n + p_{n+1})\ell_n}, e^{i(p_n + p_{n+1})\ell_{n+1}}$ coinciding ▮'s
    coeff $= 0 \Rightarrow \tilde{A}_w(\tilde{p})$
  - 'boundary' (▮'s at $\ell = 0$, $\ell = L+1$) $\Rightarrow$ BAE for $\tilde{p}$

note: same structure as for XXZ:
good: wave eq^n ~ wanted $\Rightarrow$ eigval
bad: bounce ~ internal $\Rightarrow$ coeff A
PBCs $\Rightarrow$ BAE

exc do this for $M = 1, 2$ (3)

results:

- $$\Lambda(\tilde{p}) = a^L \prod_{m=1}^{M} \frac{b(e^{ip_m}a - b) + c^2}{a(e^{ip_m}a - b)} + b^L \prod_{m=1}^{M} \frac{a(e^{ip_m}a - b) - c^2}{b(e^{ip_m}a - b)}$$

not funct^{ly} additive but still pretty simple

- $\dfrac{\tilde{A}_w(\tilde{p})}{\tilde{A}_e(\tilde{p})}$ & BAE as before

$$\underline{p} \mapsto \tilde{\underline{p}}$$
$$\Delta \mapsto \tilde{\Delta}(a,b,c) = \frac{a^2 + b^2 - c^2}{2ab}$$

given completeness, latter means

$[t(a,b,c), H_\Delta] = 0$ if $\tilde{\Delta}(a,b,c) = \Delta$!!!

$[t(a,b,c), t(a',b',c')] = 0$ if $\tilde{\Delta}(a,b,c)$
$= \tilde{\Delta}(a',b',c')$

note $(a,b,c) \mapsto (ra, rb, rc)$, $r \neq 0$
fixes $\tilde{\Delta}$, rescales Z and Λ

$\uparrow_{\deg KL}$ $\uparrow_{\deg L}$

so fix $a:b:c$ & $\tilde{\Delta}(a,b,c) = \Delta$ fixed
leaves one d.o.f., say u: $a(u), b(u), c(u)$
<u>spectral param</u>

so $t(u) = t(a(u), b(u), c(u))$

$[t(u), H_\Delta] = 0$ $\forall u$ $\left(\overset{\text{fixed}}{\Delta}\right)$!!

$[t(u), t(u')] = 0$ $\forall u, u'$

i.e. one-param fam 6vm whose transf mat comm w/ H_Δ & each other

assume $\exists$ analytic param^n of a,b,c

$\Rightarrow$ $H_k := \frac{\partial^k}{\partial u^k}\Big|_{u=u_*} \log t(u)$

$[H_k, H_\Delta] = 0$ $\forall u_* \forall k$!

$[H_k, H_{k'}] = 0$

$\tilde{\Delta}$ ''

many conserved charges! for H_Δ & 6vm@Δ_0

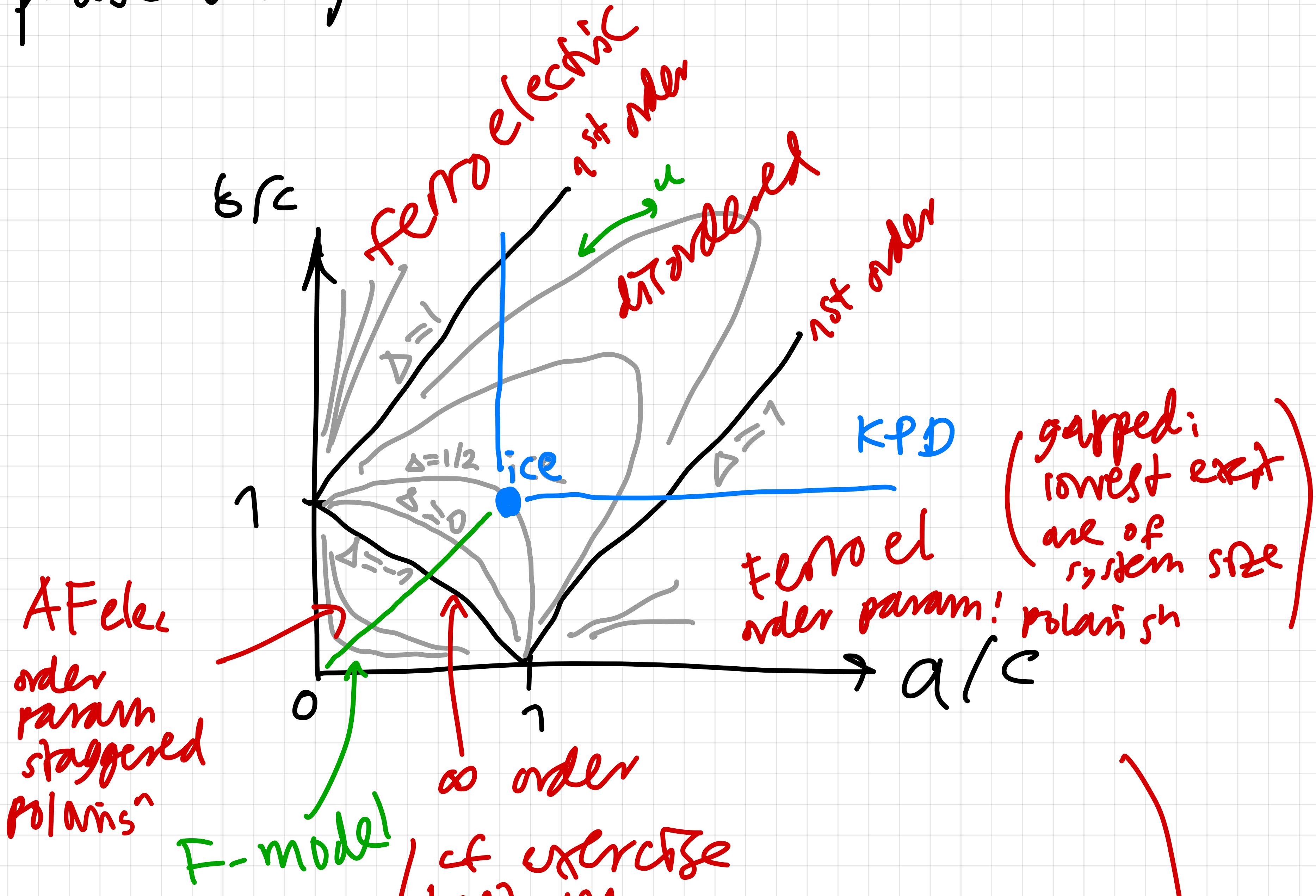
one-param fam GV

$$\begin{tikzcd} & t(u) \arrow[dl, dotted, "?"'] \arrow[dr, "u=u_0"] & \\ H_\Delta \arrow[ur, bend left, "\tilde{\Delta}=\Delta"] \arrow[rr, leftrightarrow, "symm"'] & & t(u_0) \arrow[ul, bend right, "\tilde{\Delta}=\Delta_0"'] \end{tikzcd}$$

fixed GVM

exc
$a(u) = r \sinh(u+\eta)$
$b(u) = r \sinh u$
$c(u) = r \sinh \eta$
ok; compute Δ

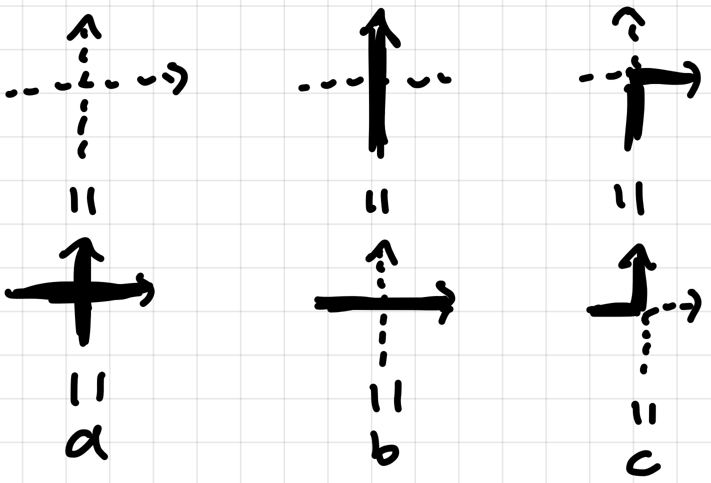
phase diagr:



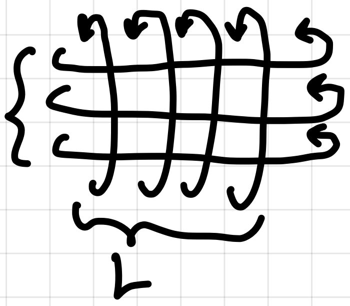
(cf exercise last time: lots of local (small) excitations on top of ground states)

recall

so far: old skool q integr



summation convention for
- internal lines
- periodic BCs

$Z = K\{$  $= \mathrm{tr}(t^K)$ cut up into horiz slices

lect 4

saw: (by CBA) $[t(u), t(u')] = 0$, $[t(u), H_\Delta] = 0$

u spectral param $\sim \{a:b:c \mid \frac{a^2+b^2-c^2}{2ab} = \Delta \text{ fixed}\}$ $\| \cosh\eta$

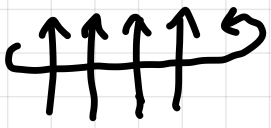
e.g. $a(u) = \rho\cdot\sinh(u+\eta)$, $b(u) = \rho\cdot\sinh(u)$, $c(u) = \rho\cdot\sinh\eta$: $\cosh\eta$

entire param; set overall normⁿ to r=1

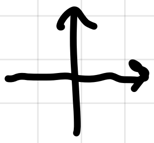
$\Delta \geq 1$: $\eta \in \mathbb{R}_{>0}$
$|\Delta| \leq 1$: $\eta = i\gamma \in i[0,\pi]$
$\Delta \leq -1$: $\eta \in i\pi + \mathbb{R}_{>0}$

⑤ quantum inverse-scattering method (QISM)

(5.1) R-matrix

$t(u) =$  attach param to horiz line

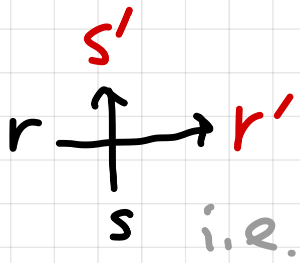
further cut up into vertical slices:

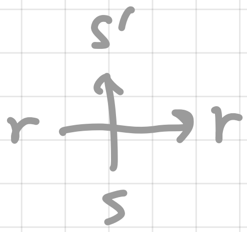
R-matrix $R(u) =$  view as operator:

$$\begin{tikzcd} \mathbb{C}^2_{\text{hor}} \otimes \mathbb{C}^2_{\text{ver}} \\ \mathbb{C}^2_{\text{hor}} \otimes \mathbb{C}^2_{\text{ver}} \arrow[u] \end{tikzcd}$$

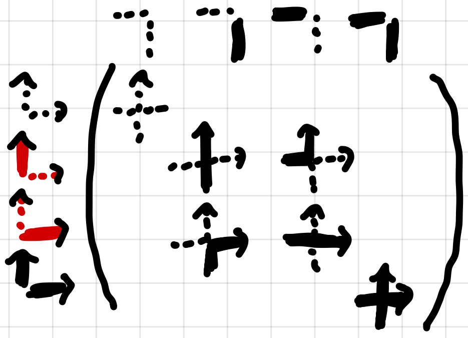
⚠

matrix elt

$\langle r', s' | R(u) | r, s \rangle =$ 

i.e. $R|r,s\rangle = \sum_{r',s'}$  $|r', s'\rangle$

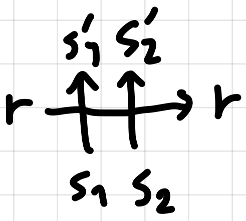
encodes vertex weights:

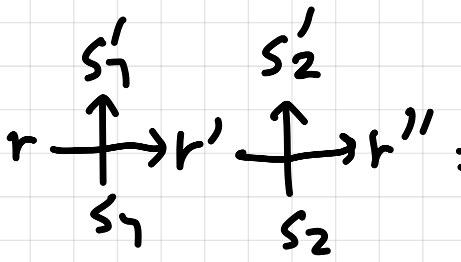
$R(u) =$  $= \begin{pmatrix} a(u) & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & a(u) \end{pmatrix}$

omitting 0's

basis ordered as $|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle$
hor ver hor ver

$|+\rangle \equiv |\uparrow\rangle$
$|-\rangle \equiv |\downarrow\rangle$



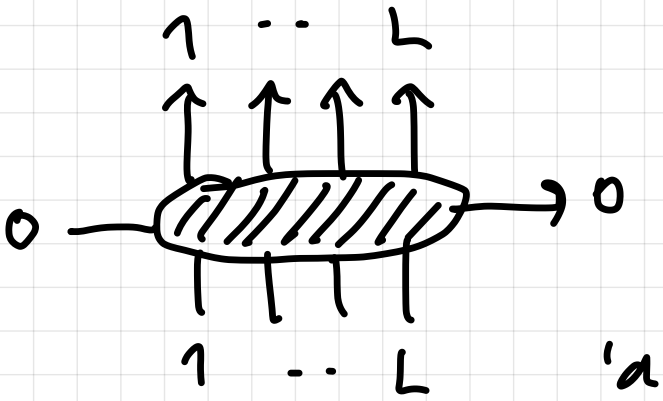
$= \sum_{r'}$ 

$= \sum_{r'} \langle r'' s_2' | R(u) | r' s_2 \rangle \langle r' s_1' | R(u) | r s_1 \rangle$

$= \sum_{r'} \langle \underset{\text{hor}}{r''} \underbrace{s_1' s_2'}_{\text{vert}} | R(u) | \underset{\text{hor}}{r'} \rangle \langle \underset{\text{hor}}{r'} | R(u) | \underset{\text{hor}}{r} \underbrace{s_1 s_2}_{\text{vert}} \rangle$

$= \langle r'' s_1' s_2' | R(u) \cdot_{\text{hor}} R(u) | r s_1 s_2 \rangle$

let's label spaces:



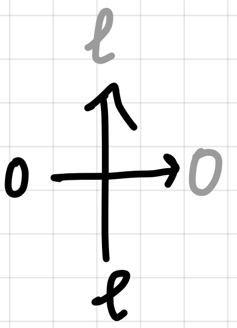
assoc ↔ $\mathbb{C}^2_0 \otimes \underbrace{\mathbb{C}^2_1 \otimes \cdots \otimes \mathbb{C}^2_L}_{\mathcal{H}}$

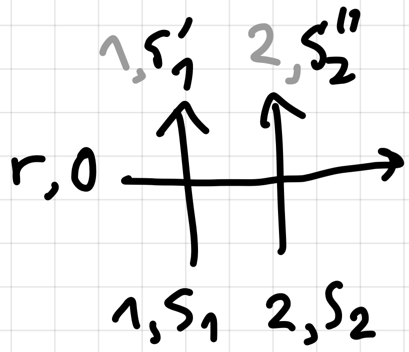
always ordered as hor ⊗ vertical: spaces follow lines

'auxiliary space' (extra from spin-chain pov)

$\mathcal{H}$ 'quantum space' (Hilb space of spin chain)

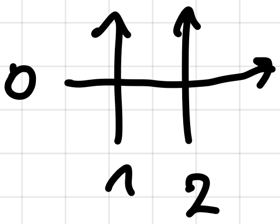
indicate spaces as subscripts on operators:

$R_{0\ell}(u) =$




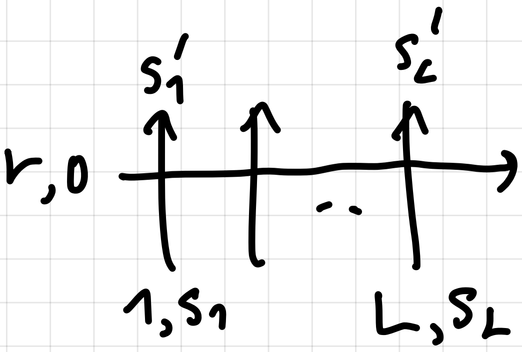
$= \langle r' s_1' s_2' | \overbrace{R_{02}}^{(u)} \overbrace{R_{01}}^{(u)} | r s_1 s_2 \rangle$

⚠ R_{01} acts first (cf orientation of lines: account for order of op's in products)

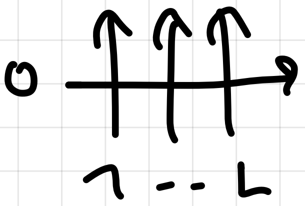
i.e.


$= R_{02}(u) R_{01}(u)$

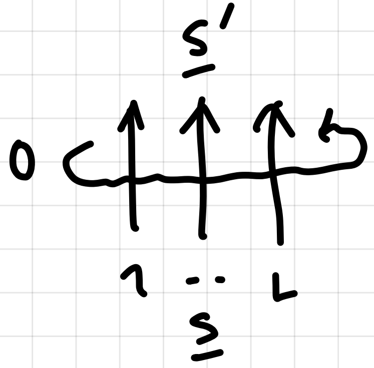
similarly

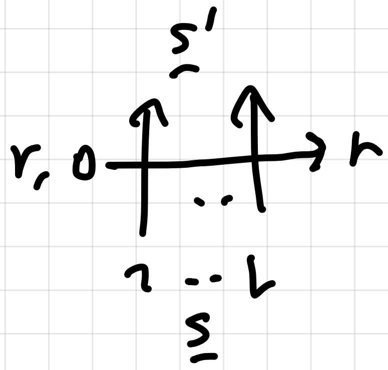


$= \langle r' s_1' \cdots s_L' | R_{0L}(u) \cdots R_{01}(u) | r s_1 \cdots s_L \rangle$

i.e.


$= R_{0L}(u) \cdots R_{01}(u)$ product of op's on $\mathbb{C}^2_0 \otimes \mathcal{H}$

$\langle \underline{s}' | t_{1..L}(u) | \underline{s} \rangle =$


$= \sum_r$


$= \sum_r \langle r \underline{s}' | R_{0L}(u) \cdots R_{01}(u) | r \underline{s} \rangle$

$= \langle \underline{s}' | \mathrm{tr}_0 (R_{0L}(u) \cdots R_{01}(u)) | \underline{s} \rangle$

in lect I left this as an exercise

i.e. $t_{1..L}(u) = \mathrm{tr}_0 (R_{0L}(u) \cdots R_{01}(u))$ trace: horizontal PBCs in terms of op^s, subscripts denote factors of $\mathbb{C}^2_0\otimes\mathcal{H} = \mathbb{C}^2_0\otimes\mathbb{C}^2_1\otimes\cdots\otimes\mathbb{C}^2_L$ on which R-matrix acts nontrivially (like for $h_{\ell\ell+1}$ in H_Δ):

$$R(u) = \sum R^{\alpha\beta}(u)\,\sigma^\alpha\otimes\sigma^\beta \mapsto R_{0\ell} = \sum R^{\alpha\beta}(u)\,\underset{0}{\sigma^\alpha}\otimes\underset{1\cdots}{\mathbb{1}^{\otimes(\ell-1)}}\otimes\underset{\ell}{\sigma^\beta}\otimes\underset{\cdots L}{\mathbb{1}^{\otimes(L-\ell)}}$$

$\sigma^0=\mathbb{1}, \sigma^\pm, \sigma^z$
basis for 2×2 mat

specifying embeddings
$\mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2)\hookrightarrow\mathrm{End}(\mathbb{C}^2_0\otimes\mathcal{H})$

and tr_0 is (partial) trace over aux space

$$\mathrm{End}(\mathbb{C}^2_0\otimes\mathcal{H})\cong\mathrm{End}(\mathbb{C}^2_0)\otimes\mathrm{End}\,\mathcal{H}\xrightarrow{\mathrm{tr}\otimes\mathrm{id}_{\mathcal{H}}}\mathbb{C}\otimes\mathrm{End}\,\mathcal{H}\cong\mathrm{End}\,\mathcal{H}$$

5.2 6vm to XXZ

recall 'trace identities' $H_k = \frac{\partial^k}{\partial u^k}\Big|_{u=u_*}\log t(u)$

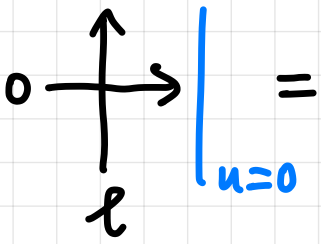
suitable special point: $R_{0\ell}(u_*)$ maximally simple

use above param^n

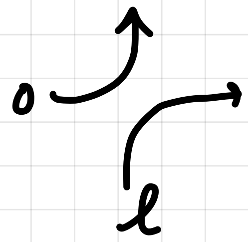
$$R(u)=\rho\cdot\begin{pmatrix}\sinh(u+\eta)&&&\\&\sinh u&\sinh\eta&\\&\sinh\eta&\sinh u&\\&&&\sinh(u+\eta)\end{pmatrix}$$

e.g. $u_*=0$: take $\rho=\frac{1}{\sinh\eta}$ for simplicity

$R_{0\ell}(0)=\cancel{\rho\sinh\eta}\,P_{0\ell}$, $P=\begin{pmatrix}1&&&\\&0&1&\\&1&0&\\&&&1\end{pmatrix}$: $P|s_1,s_2\rangle=|s_2,s_1\rangle$ spin permutation



$=\cancel{\rho\sinh\eta}$



also note $\mathrm{tr}_0 P_{0\ell} =$

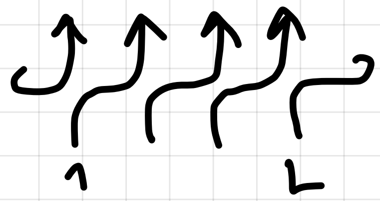
$=\mathbb{1}$

exc check using formulas

conserved charges

$t_{1\cdots L}(u_*)=$

$=\cancel{\rho\sinh\eta}$

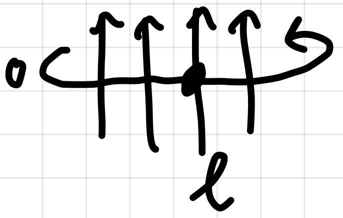
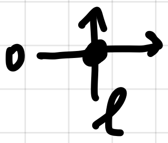


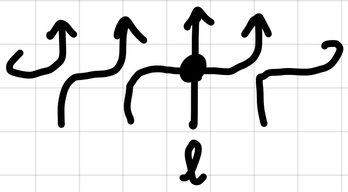
$= G^{-1}$ (right) transl

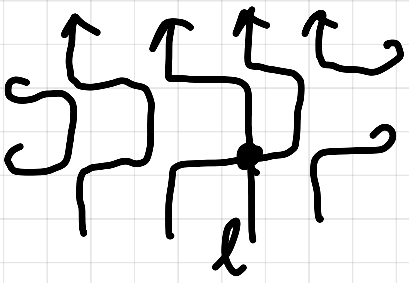
so $\log t(0) = -i\cdot$ (total) mtm for XXZ

exc do calcⁿ in formulas, check matches graphical step by step

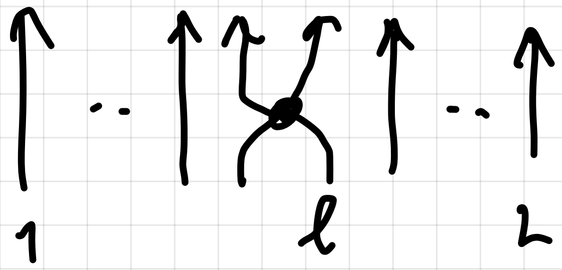
---

next: $t'(0) = \sum_{\ell=1}^{L}$  $\Big|_{u=0}$ $\qquad R'_{0\ell}(u)\Big|_{u=0} =$ 

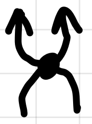
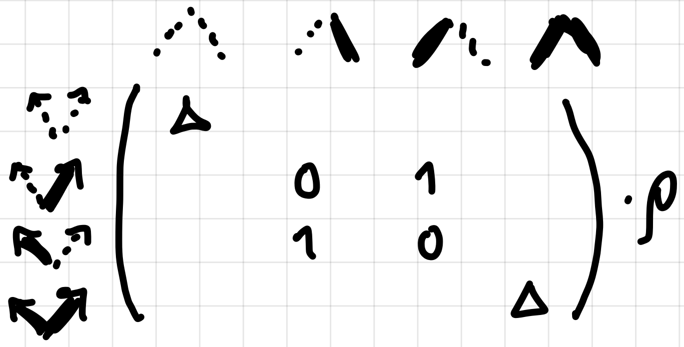
$= \cancel{(\rho \sinh \eta)^{L-1}} \sum_{\ell\in\mathbb{Z}_L}$  cf G^{-1}

so $t(0)^{-1}t'(0) = \cancel{\frac{1}{\rho \sinh \eta}} \sum_{\ell\in\mathbb{Z}_L}$ 

$\quad\;\; \|$

$\partial_u|_{u=u_*} \log t(u) \quad = \cancel{\frac{1}{\rho \sinh \eta}} \sum_{\ell\in\mathbb{Z}_L}$  ⊜

with

 $= P\,R'(0) =$ 

$$\begin{pmatrix} \Delta & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \Delta \end{pmatrix} \cdot \rho$$

$a'(0) = \rho \cosh \eta = \rho \cdot \Delta$
$b'(0) = \rho \cosh 0 = \rho$
$c'(0) = 0$

$= \cancel{\rho}\; \frac{1}{\sinh \eta}\left(\Delta \mathbb{1} \otimes \mathbb{1} + h^{\Delta}\right)$

exc check using formulas; match steps

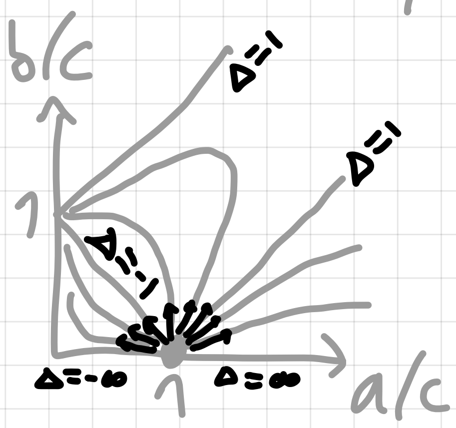
so: $\frac{1}{\rho}\, \partial_u|_{u=0} \log t(u) = \sum_{\ell\in\mathbb{Z}_L} (\Delta - h_{\ell\ell+1}) = L\Delta - H_\Delta$!!!

i.e. $\underset{\cancel{(\rho \sinh \eta)^L}}{t(u)} = G^{-1} \exp[\rho u (L\Delta - H_\Delta) + \mathcal{O}(u^2)]$

$\uparrow$ ~ infˡ genʳ
with u as "time"
(at fixed Δ; $u_*=0$)

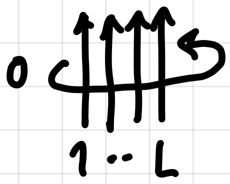
since $[t(u), t(u')] = 0$
clearly $G\, t(u)\, G^{-1} = 0$
& $[t(u), H_\Delta] = 0$

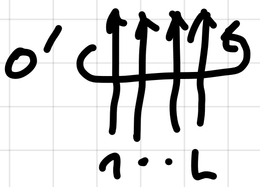
$u=0$: $a = \rho$, $b = 0$, $c = \rho$



5.3 Yang-Baxter integrability

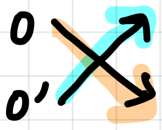
consider

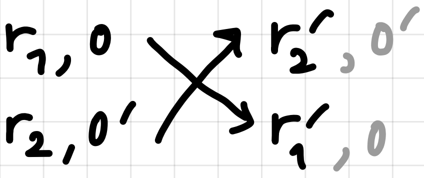
$t(u) =$  aux space $\mathbb{C}^2_0$

$t(u') =$  second aux space $\mathbb{C}^2_{0'}$

comm transf mat ⇒ lots of conserved charges ('macroscopic q integr')

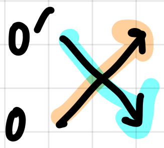
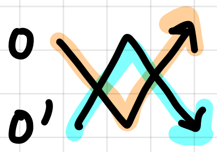
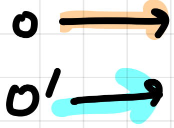
sufficient local condⁿ: Yang-Baxter q integr ('microscopic q integr')

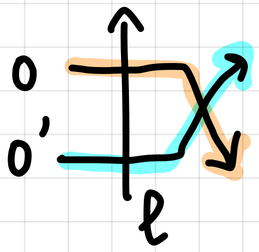
claim suppose $\exists\, R_{00'}(u,u') =$  $: \mathbb{C}^2_0 \otimes \mathbb{C}^2_{0'} \to \mathbb{C}^2_0 \otimes \mathbb{C}^2_{0'}$

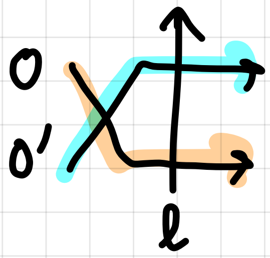
aux R-mat $\langle r_1' r_2' | R_{00'}(u,u') | r_1 r_2 \rangle =$ 

s.t.

i) generically invert (almost all u,u')

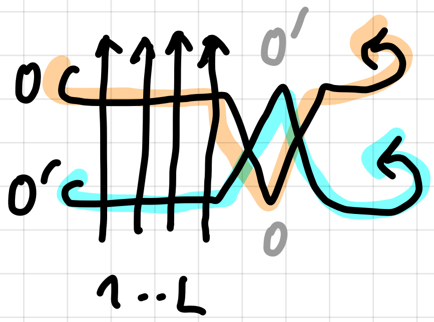
$R_{00'}(u,u')^{-1} =$  :  $=$   $=$ 

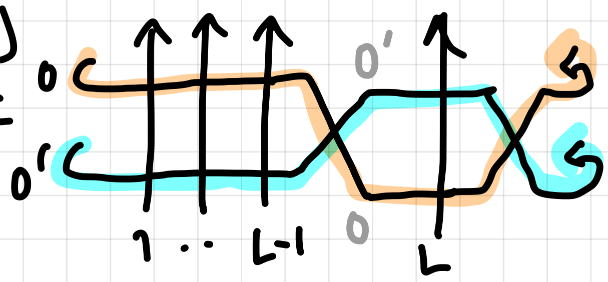
ii) Yang-Baxter equation $R_{00'}(u,u') R_{0\ell}(u) R_{0'\ell}(u') =$ 

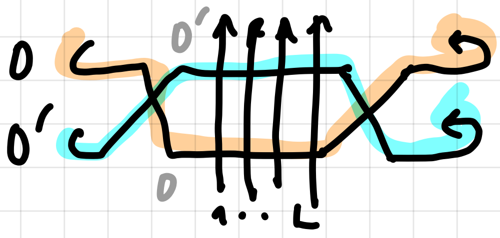
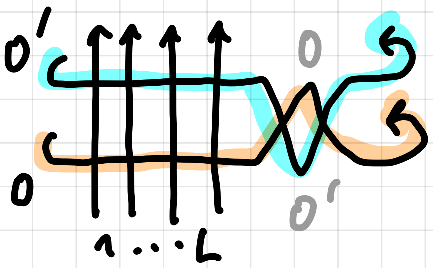
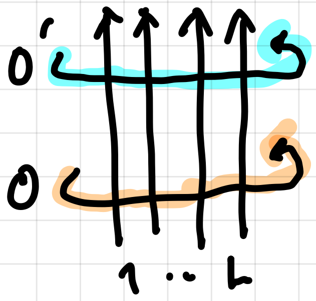
$=$  $= R_{0'\ell}(u') R_{0\ell}(u) R_{00'}(u,u')$

then $[t(u), t(u')] = 0$

exc check entrywise: diagr ⇔ eqⁿ

prf "train argument" $t(u)t(u') =$  $\overset{(i)}{=}$ 

$\overset{(ii)}{=}$  $\overset{(ii)}{=} \dots$

$\overset{(ii)}{=}$  $=$  $\overset{(i)}{=}$ 

$t(u')t(u)$

||

so $[t(u), t(u')] = 0$ for almost all u, u'

but param^n is entire, so $t(u)$ is entire (entrywise)

so $[t(u), t(u')] = 0 \quad \forall u, u'$

QED

exc check prf with formulas; match steps

to do: for given Δ, u, u' show such $R_{00'}(u,u')$ exists

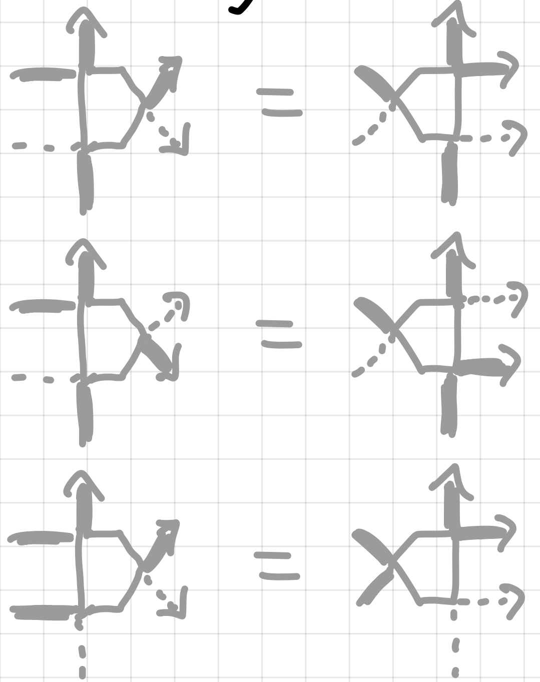
YBE: $8 \times 8 = 64$ eq^{ns} (entries)

assuming ice rule
& zero-field

$$R(u,u') = \begin{pmatrix} a'' & & & \\ & b'' & c'' & \\ & c'' & b'' & \\ & & & a'' \end{pmatrix}$$

can show YBE reduces to 3 nontriv, nonequiv^t eq^{ns}

exc use symms of diagr^s to check this



$ab'c'' + cc'b'' = \quad = \quad = ba'c''$

$ac'b'' + cb'c'' = \quad = \quad = bc'a''$

$ac'c'' + cb'b'' = \quad = \quad = ca'a''$

reobtain (w/o CBA!)

$\tilde{\Delta}(a,b,c) = \tilde{\Delta}(a',b',c')$

and $\| \tilde{\Delta}(a'',b'',c'')$

so can use same param!

$u'' = u - u'$

difference form

!!!

result: $R(u,u') = \rho'' R(u-u')$

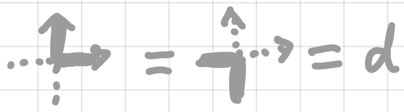
invert:

$\det R(u) = \sinh(u+\eta)^3 \sinh(u-\eta)$

iff $u'' = u - u' \neq \pm\eta + 2\pi i k, \ k \in \mathbb{Z}$

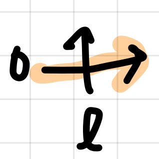
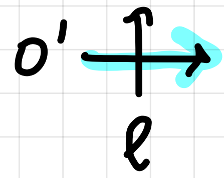
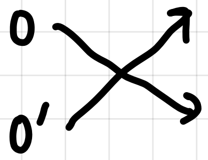
rmk only last part uses properties of vm;
claim does not

other ex: 8vm $\left[\begin{matrix} \text{find:} \\ \text{fixed} \\ \text{\& diff}^{ce} \text{ property} \end{matrix} \tilde{\Delta}(a,b,c,d) = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}, \ \tilde{\Gamma} = \frac{ab-cd}{ab+cd} \right]$

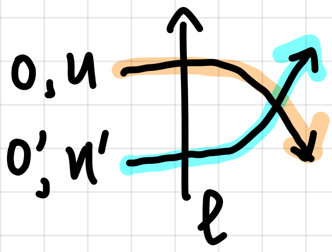
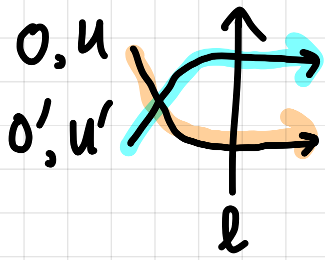


again $\exists$ entire param; claim $\Rightarrow [t, t'] = 0$;

$t(0) = G^{-1}, \ \partial_u|_{u=0} \log t = H_{\Delta,\Gamma}$ Heis XYZ

recap $R_{0\ell} =$ , $R_{0'\ell} =$ , $R_{00'} =$ 

a, b, c / ρ, u, Δ ; a', b', c' / ρ', u', Δ' ; a'', b'', c'' / ρ'', u'', Δ'' $\Big\{$ ice rule / zero field

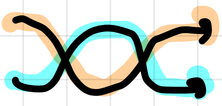
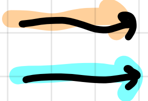
obey YBE  =  iff $\Delta = \Delta' = \Delta''$, $u'' = u - u'$

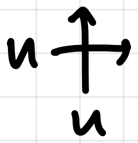
lect 5

i.e. $R_{00'}(u-u')R_{0\ell}(u)R_{0'\ell}(u')$
$= R_{0'\ell}(u')R_{0\ell}(u)R_{00'}(u-u')$

& R generic invert (*)
& $u \mapsto \langle \underline{s}'|t(u)|\underline{s}\rangle$ entries smooth $\Big\}\Rightarrow [t(u), t(u')] = 0$

---

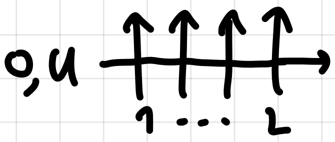
*) note: $R(u)R(-u) = a(u)a(-u)\, \mathbb{1}\otimes\mathbb{1}$  $\propto$  'unitarity'

$R(0) = a(0)\cdot P$  $\propto u$  'initial cond'

5.4 Yang-Baxter alg

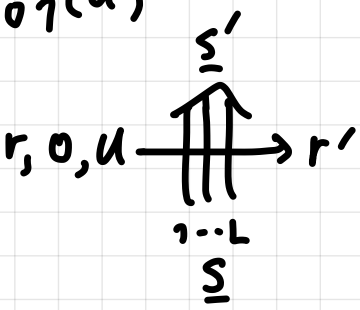
graphical short-hand

monodromy matrix
intermediate object

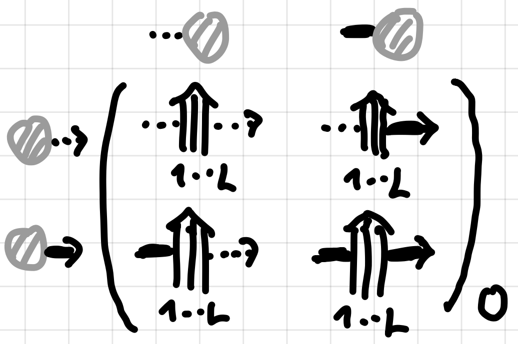
$T_{01..L}(u) =$  $=$  :

$$\begin{tikzcd} C^2_0 \otimes \mathcal{H} \\ C^2_0 \otimes \mathcal{H} \arrow[u] \end{tikzcd}$$

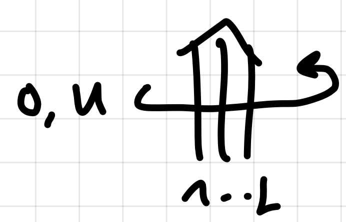
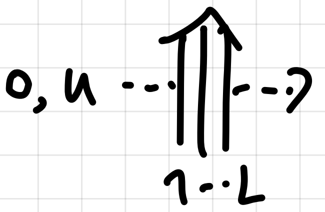
$= R_{0L}(u)\cdots R_{01}(u)$

$\langle r'\underline{s}'|T_0(u)|r\,\underline{s}\rangle =$ 

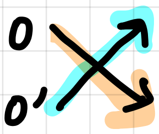
contains 4 'quantum ops':

$T_0(u) =$  $= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_0$ i.e. 2x2 mat on aux space with entries that are op's on $\mathcal{H}$

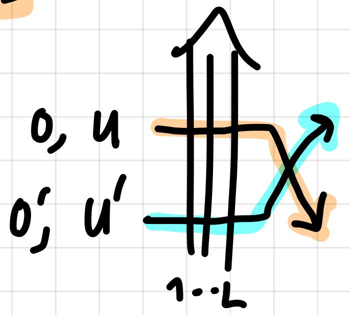
$\uparrow\uparrow\uparrow\uparrow$ 'physical content' from spin chain pov

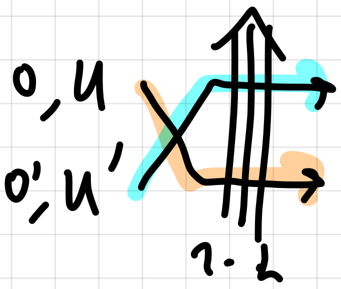
$t(u) =$  $=$  $+$ 

$= \mathrm{tr}_0 T_0(u) = A(u) + D(u)$

claim suppose $\exists R_{00'}(u,u') =$  s.t. YBE

then RTT relations

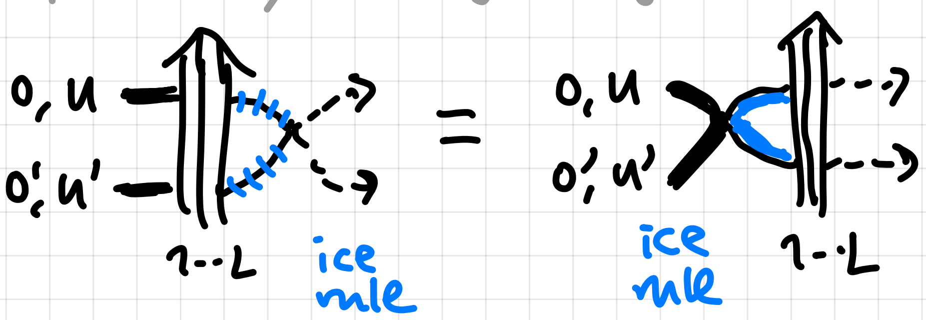
$R_{00'}(u-u')T_0(u)T_{0'}(u') =$ 

$=$  $= T_{0'}(u')T_0(u)R_{00'}(u-u')$

prf: train argument cf last time

this encodes $2^2\times 2^2=16$ rel^{ns} for $A(u),\ldots,D(u)$

ex (1,4)-entry in $\mathbb{C}^2_0\otimes\mathbb{C}^2_{0'}$



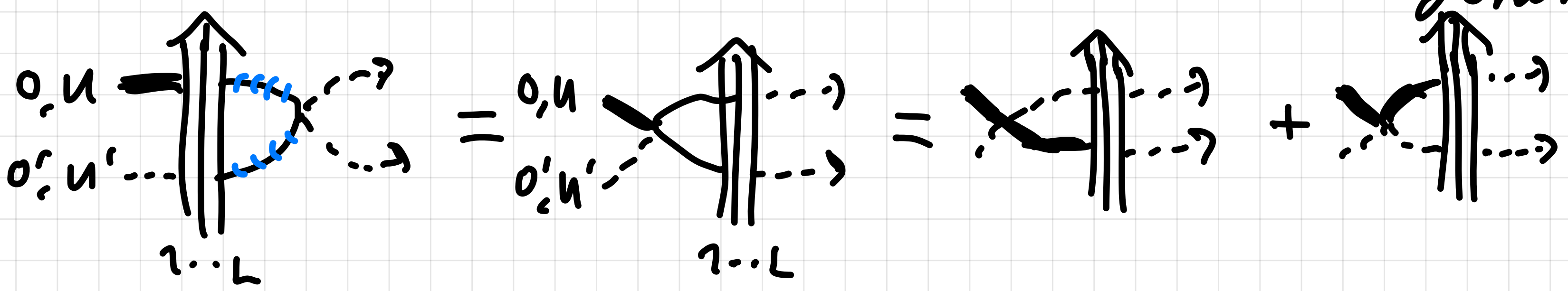
$a_+(u-u')B(u)B(u')$
$= a_-(u-u')B(u')B(u)$

i.e. $[B(u),B(u')]=0$

(provided $a(u-u')\neq 0$) cf R invert

exc check $[A(u),A(u')]=[C(u),C(u')]=[D(u),D(u')]=0$ generically

ex (1,3)-entry



$a_+(u-u')B(u)A(u') = b_-(u-u')A(u')B(u) + c_-(u-u')B(u')A(u)$

so $\boxed{A(u)B(v) = \frac{a}{b}(v-u)B(v)A(u) - \frac{c}{b}(v-u)B(u)A(v)}$ commute A through B

u ↦ v
u' ↦ u

exchange param

provided $b(u-v)\neq 0$

exc likewise show (3,4)-entry

$$\boxed{D(u)B(v) = \frac{a}{b}(u-v)B(v)D(u) - \frac{c}{b}(u-v)B(u)D(v)}$$

exc find comm rel^{ns} for $A(u), D(v)$ B, C will appear

use this to re-check $[t(u),t(v)]=0$

Yang-Baxter alg: unital assoc alg with generators $A(u), \dots, D(u)$ & RTT rel^{ns}

associative due to YBE cubic in 'struct constants' a,b,c cf Jacobi identity

		R-mat	YBA
ex	XYZ	R^{8v}	Sklyanin alg
	XXZ	R^{6v}	$U_q(L\mathfrak{sl}_2)$ quantum loop alg ("principal grading") $= U'_q(\widehat{\mathfrak{sl}_2})_{c=0}$ quantum affine alg no grading level 0
	XXX ↑ rep of ...	rat 6v $u+P$	$Y\mathfrak{sl}_2$ Yangian

'Faddeev-Reshetikhin-Takhtajan presentation'

exc in rat (XXX) case, check $T_{11}(u)=A(u)$ $T_{12}(u)=B(u)$ $T_{21}(u)=C(u)$ $T_{22}(u)=D(u)$

$$(u-v)\,[T_{ab}(u), T_{cd}(v)] = T_{cb}(v)\,T_{ad}(u) - T_{cb}(u)\,T_{ad}(v)$$

rmk really: as formal power series in u, u^{-1}

normalise $\bar{R}(u) = \dfrac{R(u)}{a(u)}$ then $A(u) = \sum_{k\geq 0} A_k u^{-k}$ etc

$(A_0 = D_0 = 1,\ B_0 = C_0 = 0)$

5.5 alg Bethe ansatz (ABA)

goal: (re)construct Bethe ansatz, i.e. eigvec of $t(u)$

recall: $[S^z, t(u)] = 0 \Rightarrow$ can do so per $M = \frac{L}{2} - S^z = \#\downarrow$'s

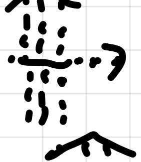
graphical: $A(u), D(u)$ preserve M

$B(u)$ injects $\downarrow$: $M \mapsto M+1$

$C(u)$ extracts $\downarrow$: $M \mapsto M-1$

so $B(u)$ is natural candidate for creation op

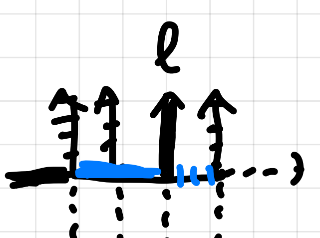
$M=0$: pseudovac $|\Omega\rangle = |\uparrow \dots \uparrow\rangle$

then $\alpha(u) := \langle\Omega|A(u)|\Omega\rangle = u$ -  $= a(u)^L$

$\delta(u) := \langle\Omega|D(u)|\Omega\rangle = u$  $= \delta(u)^L$

so $t(u)|\Omega\rangle = \Lambda_{M=0}(u)|\Omega\rangle$, $\Lambda_{M=0}(u) = \alpha(u) + \delta(u)$

$M=1$: $B(v)|\Omega\rangle$ entries:

$\langle\!\langle \ell|B(v)|\Omega\rangle = v$  $= b_-(v)^{\ell-1} c_-(v) a_+(v)^{L-\ell}$

$$= \frac{c(v)}{\delta(v)} a(v)^L \left(\frac{b(v)}{a(v)}\right)^\ell$$

vs plane waves: need $\frac{b}{a}(v) = e^{ip}$

rapidity $\lambda := -i(v+\eta/2)$: $\frac{\sin(\lambda + i\eta/2)}{\sin(\lambda - i\eta/2)} = e^{ip}$

exc check $S_\Delta(p,p') = -\frac{1-2\Delta e^{ip}+e^{i(p+p')}}{1-2\Delta e^{ip'}+e^{i(p+p')}} = \frac{\sin(\lambda-\lambda'+i\eta)}{\sin(\lambda-\lambda'-i\eta)}$ depends on difference of rapidities!

in part, XXX: $a^{rat}(v) = v+1$, $b^{rat}(v) = v$, $c^{rat}(v) = 1$:

$\lim_{\eta\to0} R(\eta u)/\eta$ ⤴ $\lambda^{rat} = -i(v+1/2) = \frac{1}{2}\cot\frac{p}{2}$ $\frac{\lambda+i/2}{\lambda-i/2} = e^{ip}$

so set $|v\rangle_B = B(v)|0\rangle$

eigenvalue: use YBA rel^n & A-, D- eigval on $|\Omega\rangle$:

$$t(u)|v\rangle_B = \Lambda_{M=1}(u;v)|v\rangle_B + \Xi(u;v)|u\rangle_B$$

$$\Lambda_{M=1}(u;v) = \frac{a}{b}(v-u)\,\alpha(u) + \frac{a}{b}(u-v)\,\delta(u)$$

$$\Xi(u;v) = -\left(\frac{c}{b}(v-u)\,\alpha(v) + \frac{c}{b}(u-v)\,\delta(v)\right)$$

$$= \frac{c}{b}(u-v)\left(\alpha(v) - \delta(v)\right)$$

$b(-u) = -b(u)$
$c(-u) = 1 = c(u)$

so $|v\rangle_B$ is eig vec w/ eig val $\Lambda_{M=1}(u;v)$

iff $1 = \frac{\delta(v)}{\alpha(v)} = \left(\frac{b(u)}{a(u)}\right)^L = e^{ipL}$ i.e. M=1 BAE! (mtm quant^n)

claim $|\underline{v}\rangle_B := B(v_1)\cdots B(v_M)|\Omega\rangle$

obeys $t(u)|\underline{v}\rangle_B = \Lambda(u;\underline{v})|\underline{v}\rangle_B$ wanted

$$+\sum_{m=1}^{M} \Xi_m(u;\underline{v})\,|\underline{v}\rangle_B\big|_{v_m\mapsto u}$$ unwanted

prf heuristics:

$$A(u)B(v_1)\cdots B(v_M) = N^A(u;\underline{v})\,B(v_1)\cdots B(v_M)A(u) + \sum_{m=1}^{M} N^A_m(u;\underline{v})\,B(v_1)\cdots B(u)\cdots B(v_M)\,A(u_m)$$

easy: $N^A(u;\underline{v}) = \prod_{m=1}^{M}\frac{a}{b}(v_m-u)$

also: $N^A_1(u;\underline{v}) = -\frac{c}{b}(v_1-u)\prod_{m=2}^{M}\frac{a}{b}(v_m-v_1)$

B's commute ⇒ LHS $=$ LHS$\big|_{v_1\leftrightarrow v_m}$

on RHS,

$$N^A(u;\underline{v}) = N^A(u;\underline{v})\big|_{v_1\leftrightarrow v_m} \quad ✓$$

$$N^A_m(u;\underline{v}) = N^A_1(u;\underline{v})\big|_{v_1\leftrightarrow v_m} = -\frac{c}{b}(v_m-u)\prod_{n(\neq m)}^{M}\frac{a}{b}(v_n-v_m)$$

exc check similar rel^n for D, with

$$N^D(u;\underline{v}) = \prod_{m=1}^{M}\frac{a}{b}(u-v_m)$$

$$N^D_m(u;\underline{v}) = -\frac{c}{b}(u-v_m)\prod_{n(\neq m)}^{M}\frac{a}{b}(v_m-v_n)$$

if we can show that $|\underline{v}\rangle_B$ are linearly indept then this is a proof. easier: use induction on M.

QED

$$\Lambda(u;\underline{v}) = N^A(u;\underline{v})\,\alpha(u) + N^D(u;\underline{v})\,\delta(u)$$

$$\Xi_m(u;\underline{v}) = N^A_m(u;\underline{v})\,\alpha(u_m) + N^D_m(u;\underline{v})\,\delta(u_m)$$

$$= \frac{c}{b}(u-v_m)\cdot BAE_m(u;\underline{v}) \Big/ \prod_{n(\neq m)}^{M} b(v_n-v_m)$$

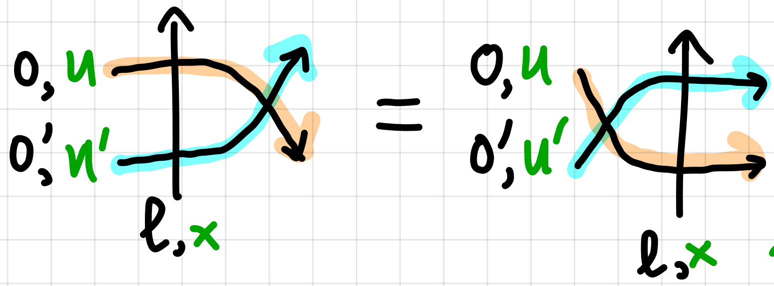
w/ $\mathrm{BAE}_m(u;v) = a(v_m)^L \prod_{n(\neq m)}^{M} a(v_n - v_m) - (-1)^{M-1} b(v_m)^L \prod_{n(\neq m)}^{M} a(v_m - v_n)$
$\overset{!}{=} 0$

up to issues of factors $=0$ iff $\frac{a}{b}(v_m)^L = (-1)^{M-1} \prod_{n(\neq m)}^{M} \frac{a(v_m - v_n)}{a(v_n - v_m)}$ BAE

<u>exc</u> check: matches $\Lambda(p)$ from §4 & BAE from §3-4

computation much easier than CBA! general M!

⑤.⑥ <u>inhomogeneities</u>

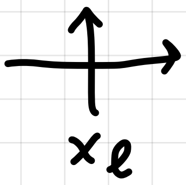


shift $u \mapsto u - x$ (still $u'' = u - u'$)
$u' \mapsto u' - x$

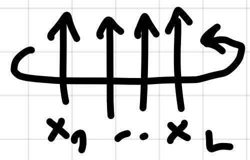
inhomog param

spectral param

diff^{ce} property suggests attaching $u \sim \mathbb{C}^2_0, u' \sim \mathbb{C}^2_{0'}, x_\ell \sim \mathbb{C}^2_\ell$

update graph not^n: $R_{0\ell}(u - x_\ell) = u$ 

use param to label spaces

$t(u;\underline{x}) = u$ 

$\xrightarrow{\ell \to 0}$ $t(u)$ homog limit

<u>exc</u> redo ABA with inhomog

<u>exc</u> check $t(x_\ell;\underline{x}) = \sinh\eta \cdot R_{\ell\ell-1}(x_\ell - x_{\ell-1}) \cdots R_{\ell 1}(x_\ell - x_1)$
$\cdot R_{\ell L}(x_\ell - x_L) \cdots R_{\ell\ell+1}(x_\ell - x_{\ell+1})$ | vs Heis XXZ? relations?

<u>exc</u> check $\frac{R(u)}{a(u)} = 1 + \eta\, r(u) + \mathcal{O}(\eta^2)$, $r(u) = \frac{1}{\sinh u}\begin{pmatrix} 0 & & & \\ & -\cosh u & 1 & \\ & 1 & -\cosh u & \\ & & & 0 \end{pmatrix}$

and find $\alpha_\ell(\underline{x}), \beta_\ell(\underline{x})$ s.t. (class^l r-mat)

$$\frac{1}{\sinh\eta} \frac{t(x_\ell;\underline{x})}{\alpha_\ell(\underline{x})} = 1 + \eta\left(H^G_\ell(\underline{x}) + \beta_\ell(\underline{x}) \cdot \mathrm{id}\right) + \mathcal{O}(\eta^2)$$

w/ $H^G_\ell(\underline{x}) = -\sum_{k(\neq\ell)}^{L} r_{k\ell}(x_k - x_\ell)$ trig

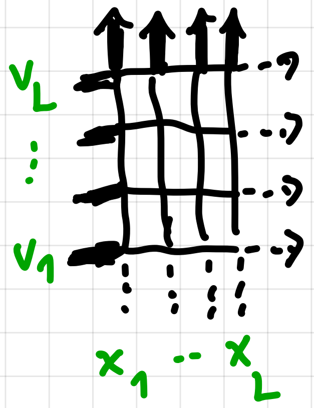
$\operatorname*{Res}_{\varepsilon=0} H^G_\ell(\varepsilon \underline{x}) = \sum_{k(\neq\ell)}^{L} \frac{1 - P_{k\ell}}{x_k - x_\ell}$ rat

} Gaudin model

application to combinatorics:

exc read Kuperberg's arXiv: math/9712207 exact & explicit

$Z_L^{DW}(\underline{v};\underline{x}) =$



⑥ long-range q integr (teaser)

isotropic level ($\Delta = 1$):

$$H = -\sum_{k<l}^{L} V(k-l)\,(1-P_{kl})$$

	Heis XXX	Inozemtsev	Haldane-Shastry
$V(d)$	$\delta_{d \bmod L,\, L-1} \xleftarrow[\kappa\to\infty]{}$	$\sim \wp(d)$ periods $L,\ w = \frac{i\pi}{\kappa}$	$\xrightarrow[\kappa\to 0]{}$ $\frac{1}{4\sin^2(\frac{\pi}{L}d)}$
	paradigm for q int	challenges our understanding of q int	paradigm for long-range q int
conserved charges	transf mat $\uparrow$	proposal, partial prf, new results	✓ (qdet; transf mat)
Yangian	monodromy $\downarrow$ ABA	?? not known!	✓ (different rep)
exact eig vec	Bethe vec (up to solving BAE)	'extended CBA' (conn to ell CS; up to solving BAE)	explicit (w/ Jack pol: no BAE conn to eig CS)